(Elasticity)

Self-consistent scheme

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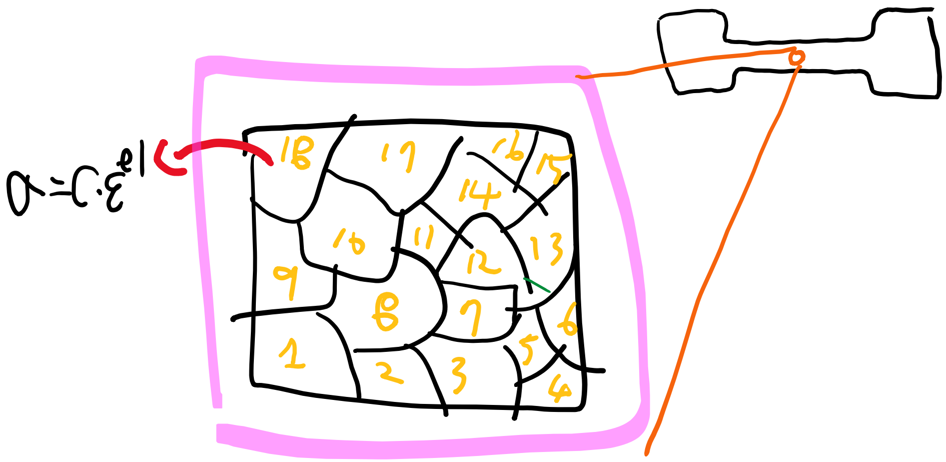
The entire response/stimulus and individual response/stimulus?



Assumptions

* We know moduli/compliance of each grain.
* And let's assume they are "uniform" *
* Each member has "uniform" weight *

* These are not "REQUIRED", but will make our analyses simple.

Two extreme cases

1. $\bar{C}^{el} = \langle C^{el} \rangle = \langle (M^{el})^{-1} \rangle$

Thus, $\bar{M}^{el} = (\bar{C}^{el})^{-1}$; inverse

2. $\bar{M}^{el} = \langle M^{el} \rangle$

Thus, $\bar{C}^{el} = (\bar{M}^{el})^{-1} = (\langle M^{el} \rangle)^{-1}$

* The above two cases may lead to "equivalent" results OR NOT !

$$\bar{\varepsilon}^{el} = \bar{M}^{el} \cdot \bar{\sigma}$$

$$\boxed{\bar{\varepsilon}^{el} = \langle M^{el} \rangle \bar{\sigma}}$$

or

$$\boxed{\langle C^{el} \rangle \bar{\varepsilon}^{el} = \bar{\sigma}}$$

$$\bar{\varepsilon}^{el} = \langle C^{el} \rangle^{-1} \bar{\sigma}$$

case 2

case 1

$$\bar{\varepsilon}^{el} = \left(\langle C^{el} \rangle^{-1} \right) \bar{\sigma}$$

case 2, if

$$\sigma = \bar{\sigma}$$

$$\bar{\varepsilon}^{el} = \langle \varepsilon^{el} \rangle$$

for case #1, if $\varepsilon^{el} = \bar{\varepsilon}^{el}$,

$$\left(C^{el,1} \omega^1 + C^{el,2} \omega^2 + C^{el,2} \omega^2 \right) \cdot \varepsilon^{el} = \bar{\sigma}$$

$$\rightarrow C^{el,1} \varepsilon^{el,1} \omega^1 + C^{el,2} \varepsilon^{el,2} \omega^2 + C^{el,3} \varepsilon^{el,3} \omega^3$$

$$\rightarrow \langle \sigma \rangle = \bar{\sigma}$$

Neither of the assumptions is realistic.

→ Due to the INTERACTION between members, the stress or strain should be "inhomogeneous".

$$\varepsilon \neq \bar{\varepsilon}$$

$$\sigma \neq \bar{\sigma}$$

self - consistent condition

$\bar{\sigma} = \langle \sigma \rangle$

and

$\bar{\varepsilon}^{el} = \langle \varepsilon^{el} \rangle$

we need

$$\bar{\sigma} = \bar{C}^{el} \bar{\varepsilon}^{el}$$

$$\bar{\varepsilon}^{el} = \bar{M}^{el} \bar{\sigma}$$

$$\bar{C}^{el} = (\bar{M}^{el})^{-1}$$

At the same time we know

$$\sigma = C^{el} \varepsilon^{el}$$

$$\varepsilon^{el} = M^{el} \sigma$$

$$C^{el} = (M^{el})^{-1}$$

Now, the question is how to obtain $\bar{C}^{el}$ by C^{el}

In other word, how to use "lower scale" property to estimate "upper scale" property?

Self-Consistent scheme

J. Mech. Phys. Solids, 1965, Vol. 13, pp. 213 to 222. Pergamon Press Ltd. Printed in Great Britain.

A SELF-CONSISTENT MECHANICS OF COMPOSITE MATERIALS

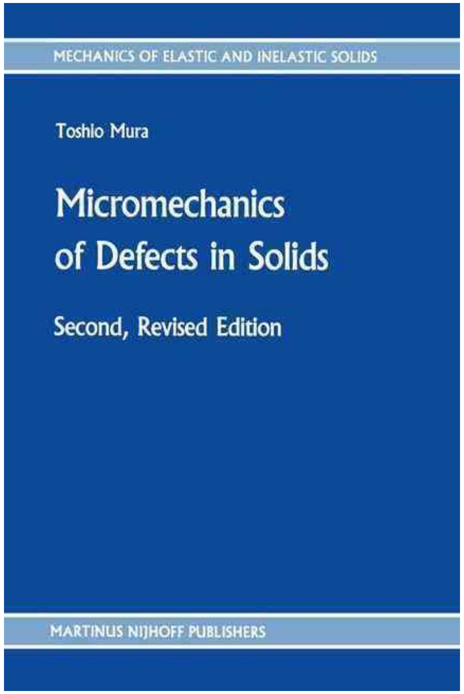
By R. Hill
Department of Applied Mathematics and Theoretical Physics, University of Cambridge

In association with Eshelby's analysis on elastic inclusion in HEM

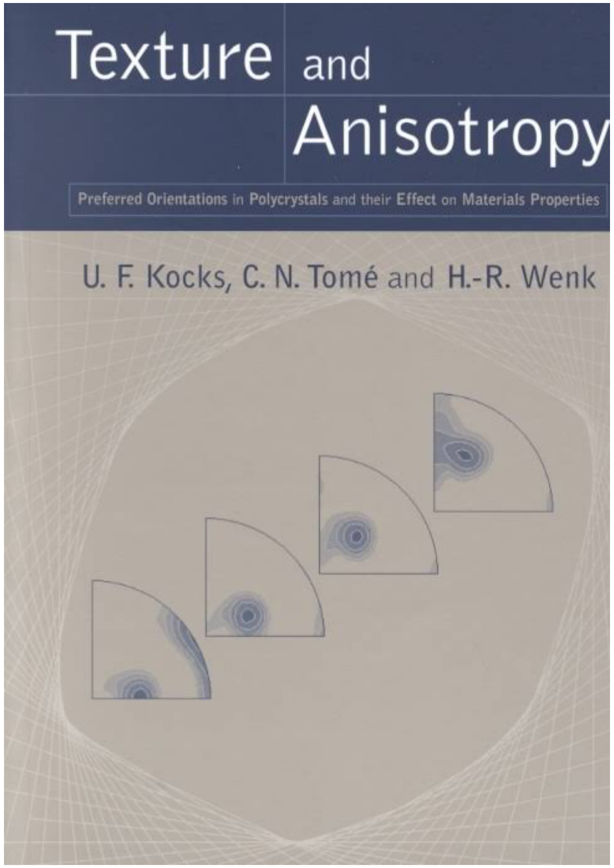
The determination of the elastic field of an ellipsoidal inclusion, and related problems

By J. D. Eshelby
Department of Physical Metallurgy, University of Birmingham

Another good reference to look at



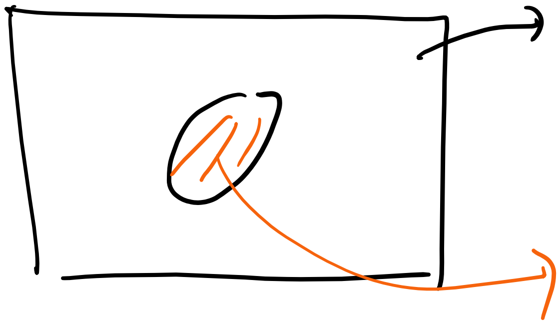
→ The go-to book for those wish to study this subject more seriously.


Texture and Anisotropy
Preferred Orientations in Polycrystals and their Effect on Materials Properties
U. F. Kocks, C. N. Tomé and H.-R. Wenk

What I'd suggest to my dear students :

* Do not attempt to understand the mathematical derivations presented in the books.
  - To understand the book you need to be familiar with advanced level of mathematical backgrounds.
* Instead, here in my lecture, I'd present the use of mathematical findings.
  specific to our problem, (physical)

What these findings are ...



homogeneous equivalent medium. (HEM)
(infinitely large).

inclusion in spherical shape

* We can figure out the neighbouring effect (interaction btwn inclusion & medium) by the use of "Eshelby" tensor.
* This Eshelby tensor tells you relative stiffness of the medium towards the inclusion
* Eshelby tensor is function of
  - inclusion shape
  - HEM's property (modulus ..)

* Elastic inclusion in elastic HEM.

- for each grain

$$\varepsilon^{el} = M^{el} \cdot \sigma \quad ; \text{ Hooke's law.}$$

in tensorial quantities written in Matrix form:

$$\varepsilon^{el}_{ij} = M^{el}_{ijkl} \sigma_{kl}$$

- for elastic HEM

$$\bar{\varepsilon}^{el}_{ij} = \bar{M}^{el}_{ijkl} \bar{\sigma}_{kl}$$

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we are looking at history (evolution) of material w.r.t time.

$$\therefore \quad \dot{\varepsilon}^{el}_{ij} = M^{el}_{ijkl}\,\dot{\sigma}_{kl}$$

$$\dot{\bar{\varepsilon}}^{el}_{ij} = \bar{M}^{el}_{ijkl}\,\dot{\bar{\sigma}}_{kl}$$

$$\dot{\square} = \frac{d\square}{dt}$$

The gist of Eshelby's result:

$$\boxed{\left(\dot{\varepsilon}^{el}_{ij} - \dot{\bar{\varepsilon}}^{el}_{ij}\right) = -\,\tilde{M}^{el}_{ijkl}\left(\dot{\sigma}_{kl} - \dot{\bar{\sigma}}_{kl}\right)}$$

$$\tilde{M}^{el}_{ijop} = \left(I_{ijkl} - S^{el}_{ijkl}\right)^{-1} S^{el}_{klmn}\,\bar{M}^{el}_{mnop}$$

$\bar{M}^{el}$; unknown priori → let's assume we know this somehow.

self consistent condition ?

$$\boxed{\begin{aligned}\langle \dot{\varepsilon}^{el} \rangle &= \bar{\dot{\varepsilon}}^{el} \\ \langle \dot{\sigma} \rangle &= \bar{\dot{\sigma}}\end{aligned}}$$

if $\bar{M}^{el}$ is truely "self-consistent" → this should be satisfied.

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let's substitute this to interaction equation.

$$\dot{\varepsilon}^{el}_{ij} - \bar{\dot{\varepsilon}}^{el}_{ij} = -\tilde{M}^{el}_{ijkl}\dot{\sigma}_{kl} + \tilde{M}_{ijkl}\bar{\dot{\sigma}}_{kl}$$

$$M^{el}_{ijkl}\dot{\sigma}_{kl} - \bar{M}_{ijkl}\bar{\dot{\sigma}}_{kl} = -\tilde{M}^{el}_{ijkl}\dot{\sigma}_{kl} + \tilde{M}^{el}_{ijkl}\bar{\dot{\sigma}}_{kl}$$

$$\Rightarrow (M^{el}_{ijkl} + \tilde{M}^{el}_{ijkl}) \cdot \dot{\sigma}_{kl} = (\bar{M}_{ijkl} + \tilde{M}^{el}_{ijkl})\bar{\dot{\sigma}}_{kl}.$$

→ If we rearrange it, we'll get

$$\dot{\sigma}_{ij} = \underbrace{\left(M^{el}_{ijkl} + \tilde{M}^{el}_{ijkl}\right)^{-1} \left(\bar{M}^{el}_{klmn} + \tilde{M}^{el}_{klmn}\right)}_{\text{this gives} \rightsquigarrow B_{ijmn}} \dot{\bar{\sigma}}_{mn}$$

$$\dot{\sigma}_{ij} = \underbrace{B_{ijkl}}_{\text{function of } M^{el}_{ijkl},\ \bar{M}^{el}_{ijkl},\ \tilde{M}^{el}_{ijkl}} \dot{\bar{\sigma}}_{kl}.$$

$\tilde{M}^{el}_{ijkl}$ → function of $\bar{M}_{ijkl}$, inclusion shape

→ B_{ijkl} function of ① M^{el}_{ijkl} ② $\bar{M}^{el}_{ijkl}$ ③ shape

we don't know (initially) → unknown priors

$$\bar{\dot{\varepsilon}} = \langle \dot{\varepsilon} \rangle$$

$$\dot{\varepsilon}^{el} = M^{el} \dot{\sigma}$$

$$\bar{\dot{\varepsilon}}^{el} = \bar{M}^{el} \bar{\dot{\sigma}}$$

$$\bar{M}^{el} \bar{\dot{\sigma}} = \langle M^{el} \dot{\sigma} \rangle = \langle M^{el} \cdot \underline{\beta \cdot \bar{\dot{\sigma}}} \rangle$$

$$\therefore \bar{M}^{el} \bar{\dot{\sigma}} = \langle M^{el} \cdot \beta \rangle \cdot \bar{\dot{\sigma}}$$

$$\therefore \bar{M}^{el} = \langle M^{el} \cdot \beta \rangle$$

* We start with guessing $\bar{M}^{el,(1st)}$

1st guess.

i) → calculate Eshelby tensor $S_{ijkl}^{el\,(1st)}$

ii) → calculate $\tilde{M}_{ijkl}^{el\,(1st)}$

iii) → calculate B_{ijkl}

iv) → One can back-calculate $\bar{M}^{el}$ from

$$\bar{M}^{el,(2nd)} = \langle M^{el} \cdot B \rangle$$ new

→ we iteratively estimate $\boxed{\bar{M}^{el}}$

* matrix.

- Addition

$$(A+B)_{ij} = A_{ij} + B_{ij}$$

- scalar multiplication

$$(cA)_{ij} = c \cdot A_{ij}$$

- transposition

$$(A^T)_{ij} = A_{ji}$$

- Matrix multi