Self-consistent scheme

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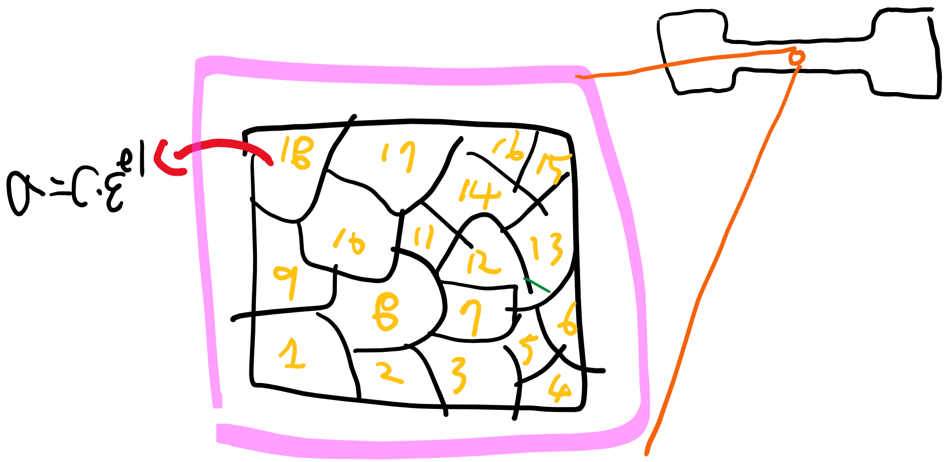
The entire response/stimulus and individual response/stimulus?



Assumptions

* We know moduli/compliance of each grain.
* And let's assume they are "uniform" *
* Each member has "uniform" weight *

* These are not "REQUIRED", but will make our analyses simple.

Two extreme cases

1. $\bar{C}^{el} = \langle C^{el} \rangle = \langle (M^{el})^{-1} \rangle$

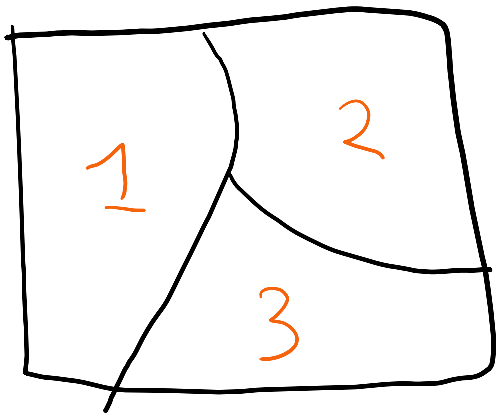
Thus, $\bar{M}^{el} = (\bar{C}^{el})^{-1}$; inverse

2. $\bar{M}^{el} = \langle M^{el} \rangle$

Thus, $\bar{C}^{el} = (\bar{M}^{el})^{-1} = (\langle M^{el} \rangle)^{-1}$

* The above two cases may lead to "equivalent" results OR NOT !

Let's use aggregate with only 3 members.



$w^1 = \frac{1}{3}$

$w^2 = \frac{1}{3}$

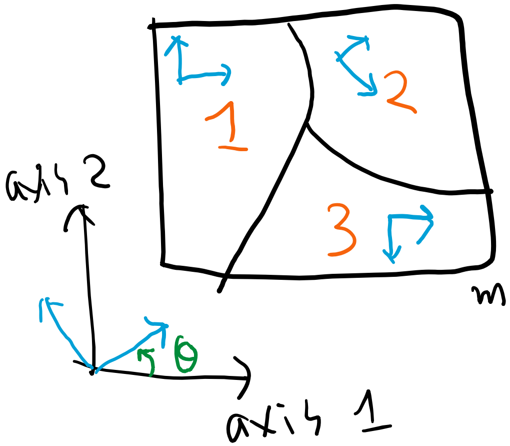
$w^3 = \frac{1}{3}$

* Each member has the same weights.

Although our aggregate may seem overly simplified, things we'll learn from it can be applied to

* Other aggregates with many more members.
* Aggregate with non equal weights

Isotropic members but with tensors



$w^1 = \frac{1}{3}$

$w^2 = \frac{1}{3}$

$w^3 = \frac{1}{3}$

*Each member has the same weights.

member 1, 2, 3

$\theta = 0°, -45°, -90°$

$w = \frac{1}{3}, \frac{1}{3}, \frac{1}{3}$

coordinate transformation matrix.

$R_{ij}^{member \leftarrow aggregate}$

ξelln, $R_{ij}^{m\leftarrow a}$

member 1

$$R_{ij}^{m\leftarrow a} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

member 2

$$R_{ij}^{m\leftarrow a} = \begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

member 3

$$R_{ij}^{m\leftarrow a} = \begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Note transformation Rule.

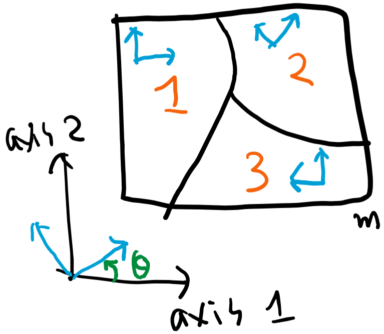
$$\sigma'_i = R_{ij}\sigma_j$$

$$\varepsilon^{el'}_i = R_{ij}\varepsilon^{el}_j$$

$$C^{el'}_{ij} = R_{ik}R_{jl}C^{el}_{kl}$$

$$M^{el'}_{ij} = R_{ik}R_{jl}C^{el}_{kl}$$

Isotropic members



$w^1 = \frac{1}{3}$

$w^2 = \frac{1}{3}$

$w^3 = \frac{1}{3}$

* Each member has the same weights.

member	1	2	3
$\theta =$	$0°$	$45°$	$90°$
$w =$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

$$C^{el}_{ij} = \begin{bmatrix} 200 & 0 \\ 0 & 200 \end{bmatrix} [\text{GPa}]$$

Note transformation Rule.

$$\sigma'_i = R_{ij}\sigma_j$$

$$\varepsilon^{el'}_i = R_{ij}\varepsilon^{el}_j$$

$$C^{el'}_{ij} = R_{ik}R_{jl}C^{el}_{kl}$$

$$M^{el'}_{ij} = R_{ik}R_{jl}C^{el}_{kl}$$

member 1.

$$C^{el,'}_{ij} = \overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}^{R} \begin{bmatrix} 200 & 0 \\ 0 & 200 \end{bmatrix} \overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}^{R^T} = \boxed{\begin{bmatrix} 200 & 0 \\ 0 & 200 \end{bmatrix} \text{GPa}}$$

member 2.

$$C^{el,'}_{ij} = \overbrace{\frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}^{R} \begin{bmatrix} 200 & 0 \\ 0 & 200 \end{bmatrix} \overbrace{\frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}^{R^T} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 200 & 0 \\ 0 & 200 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 200 & -200 \\ 200 & 200 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 400 & 0 \\ 0 & 400 \end{bmatrix} = \boxed{\begin{bmatrix} 200 & 0 \\ 0 & 200 \end{bmatrix} \text{GPa}}$$

member 3.

$$C^{el,'}_{ij} = \overbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}^{R} \begin{bmatrix} 200 & 0 \\ 0 & 200 \end{bmatrix} \overbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}^{R^T} = \begin{bmatrix} 0 & -200 \\ 200 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 200 & 0 \\ 0 & 200 \end{bmatrix} \text{GPa}}$$

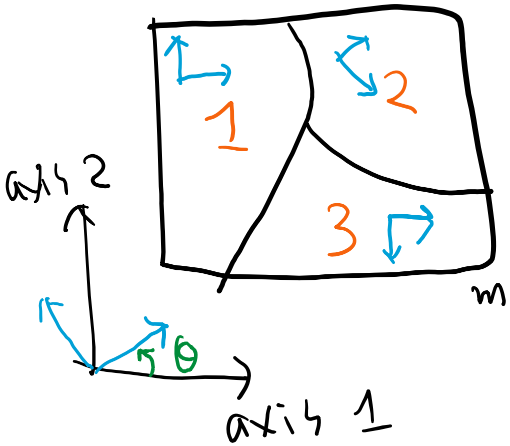
Isotropic property was assumed for each grain

→ Isotropic property does not depend on coordinate system.

→ Even if grains with different orientation, it eventually reduces the case of "unifrom" grain property.

→ Thus, the two extreme cases would lead to the same results,

Anisotropic members



$w^1 = \frac{1}{3}$

$w^2 = \frac{1}{3}$

$w^3 = \frac{1}{3}$

* Each member has the same weights.

member	1,	2,	3
$\theta =$	$0°$,	$-45°$,	$-90°$
$w =$	$\frac{1}{3}$,	$\frac{1}{3}$,	$\frac{1}{3}$

coordinate transformation matrix.

$R_{ij}^{member \leftarrow aggregate}$

줄이면, $R_{ij}^{m \leftarrow a}$

member 1

$$R_{ij}^{m\leftarrow a} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

member 2

$$R_{ij}^{m\leftarrow a} = \begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

member 3

$$R_{ij}^{m\leftarrow a} = \begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Note transformation Rule.

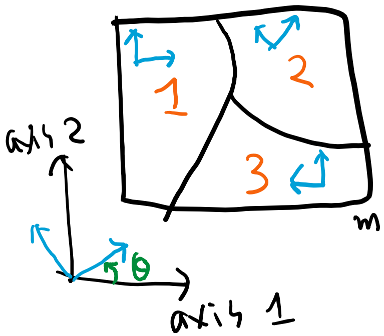
$$\sigma_i' = R_{ij}\sigma_j$$

$$\varepsilon^{el\prime}_i = R_{ij}\varepsilon^{el}_j$$

$$C^{el\prime}_{ij} = R_{ik}R_{jl}C^{el}_{kl}$$

$$M^{el\prime}_{ij} = R_{ik}R_{jl}C^{el}_{kl}$$

Anisotropic members



$w^1 = \frac{1}{3}$
$w^2 = \frac{1}{3}$
$w^3 = \frac{1}{3}$

* Each member has the same weights.

member 1, 2, 3

$\theta = 0°, 45°, 90°$

$w = \frac{1}{3}, \frac{1}{3}, \frac{1}{3}$

$$C^{el}_{ij} = \begin{bmatrix} 200 & 0 \\ 0 & 100 \end{bmatrix} [GPa]$$

Note transformation Rule.

$$\sigma'_i = R_{ij}\sigma_j$$
$$\varepsilon^{el,'}_i = R_{ij}\varepsilon^{el}_j$$
$$C^{el,'}_{ij} = R_{ik}R_{jl}C^{el}_{kl}$$
$$M^{el,'}_{ij} = R_{ik}R_{jl}c^{el}_{kl}$$

member 1.

$$C^{el,'}_{ij} = \overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}^{R}\begin{bmatrix} 200 & 0 \\ 0 & 100 \end{bmatrix}\overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}^{R^T} = \boxed{\begin{bmatrix} 200 & 0 \\ 0 & 100 \end{bmatrix} GPa}$$

member 2.

$$C^{el,'}_{ij} = \overbrace{\frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}^{R}\begin{bmatrix} 200 & 0 \\ 0 & 100 \end{bmatrix}\overbrace{\frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}^{R^T} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 200 & 0 \\ 0 & 100 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 200 & -100 \\ 200 & 100 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 300 & 100 \\ 100 & 300 \end{bmatrix} = \boxed{\begin{bmatrix} 150 & 50 \\ 50 & 150 \end{bmatrix} GPa}$$

member 3.

$$C^{el,'}_{ij} = \overbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}^{R}\begin{bmatrix} 200 & 0 \\ 0 & 100 \end{bmatrix}\overbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}^{R^T} = \begin{bmatrix} 0 & -100 \\ 200 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 100 & 0 \\ 0 & 200 \end{bmatrix} GPa}$$

We are going to explore two options to obtain $\bar{\xi}^{el}$ by estimating the compliance of aggregate $\bar{M}^{el}$ using

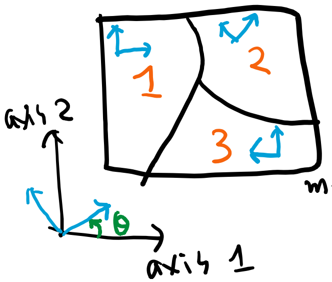
$$\bar{M}^{el} = \left(\bar{C}^{el}\right)^{-1} = \left(\langle C^{el} \rangle \right)^{-1}$$

OR

$\langle\ \rangle$: weight average.

$$\bar{M}^{el} = \langle M^{el} \rangle = \left\langle \left(C^{el}\right)^{-1} \right\rangle$$

Anisotropic members



$w^1 = \frac{1}{3}$

$w^2 = \frac{1}{3}$

$w^3 = \frac{1}{3}$

*Each member has the same weights.

member 1, 2, 3

$\theta = 0°, 45°, 90°$

$w = \frac{1}{3}, \frac{1}{3}, \frac{1}{3}$

$$C^{el}_{ij} = \begin{bmatrix} 200 & 0 \\ 0 & 100 \end{bmatrix} [GPa]$$

Note transformation Rule.

$\sigma'_i = R_{ij}\sigma_j$

$\varepsilon^{el'}_i = R_{ij}\varepsilon^{el}_j$

$C^{el'}_{ij} = R_{ik}R_{jl}C^{el}_{kl}$

$M^{el'}_{ij} = R_{ik}R_{jl}C^{el}_{kl}$

Extreme case 1

$$\bar{C}^{el}_{ij} = \langle C^{el}_{ij} \rangle = \begin{bmatrix} 200 & 0 \\ 0 & 100 \end{bmatrix}\frac{1}{3} + \begin{bmatrix} 150 & 50 \\ 50 & 150 \end{bmatrix}\frac{1}{3} + \begin{bmatrix} 100 & 0 \\ 0 & 200 \end{bmatrix}\frac{1}{3}$$

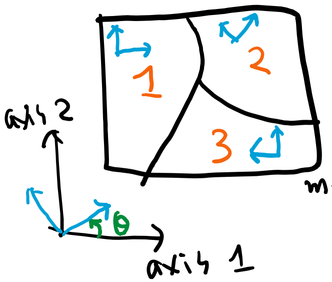
$$= \begin{bmatrix} 450 & 50 \\ 50 & 450 \end{bmatrix}\frac{1}{3} = \begin{bmatrix} 9 & 1 \\ 1 & 9 \end{bmatrix}\frac{50}{3}$$

$\bar{M} = (\bar{C})^{-1}$

$$\therefore \boxed{\bar{M}^{el}_{ij}} = \frac{1}{81-1}\begin{bmatrix} 9 & -1 \\ -1 & 9 \end{bmatrix}\cdot\frac{3}{50} = \frac{3}{80\cdot 50}\begin{bmatrix} 9 & -1 \\ -1 & 9 \end{bmatrix}$$

$$= \frac{3}{4000}\begin{bmatrix} 9 & -1 \\ -1 & 9 \end{bmatrix} = \frac{3}{4000}\begin{bmatrix} 9 & -1 \\ -1 & 9 \end{bmatrix} = \frac{1}{1000}\begin{bmatrix} \frac{27}{4} & -\frac{3}{4} \\ -\frac{3}{4} & \frac{27}{4} \end{bmatrix} = \boxed{\frac{1}{1000}\begin{bmatrix} 6.75 & -0.75 \\ -0.75 & 6.75 \end{bmatrix}}$$

Anisotropic members



$w^1 = \frac{1}{3}$

$w^2 = \frac{1}{3}$

$w^3 = \frac{1}{3}$

*Each member has the same weights.

member 1, 2, 3

$\theta = 0°, 45°, 90°$

$w = \frac{1}{3}, \frac{1}{3}, \frac{1}{3}$

$$C^{el}_{ij} = \begin{bmatrix} 200 & 0 \\ 0 & 100 \end{bmatrix} [GPa]$$

Note transformation Rule.

$\sigma'_i = R_{ij}\sigma_j$

$\varepsilon^{el'}_i = R_{ij}\varepsilon^{el}_j$

$C^{el'}_{ij} = R_{ik}R_{jl}C^{el}_{kl}$

$M^{el'}_{ij} = R_{ik}R_{jl}C^{el}_{kl}$

Extreme case 2

member1 $\quad C^{el}_{ij} = \begin{bmatrix} 200 & 0 \\ 0 & 100 \end{bmatrix} \rightarrow M^{el}_{ij} = \frac{1}{20000}\begin{bmatrix} 100 & 0 \\ 0 & 200 \end{bmatrix} = \begin{bmatrix} \frac{1}{200} & 0 \\ 0 & \frac{1}{100} \end{bmatrix}$

member2 $\quad C^{el}_{ij} = \begin{bmatrix} 150 & 50 \\ 50 & 150 \end{bmatrix} \rightarrow M^{el}_{ij} = \frac{1}{150^2 - 50^2}\begin{bmatrix} 150 & -50 \\ -50 & 150 \end{bmatrix} = \frac{1}{200 \times \cancel{100}}\begin{bmatrix} 15\cancel{0} & -5\cancel{0} \\ -5\cancel{0} & 15\cancel{0} \end{bmatrix}$

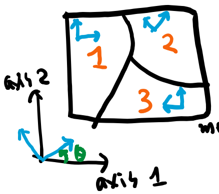
$\longrightarrow \frac{1}{400}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$

member3 $\quad C^{el}_{ij} = \begin{bmatrix} 100 & 0 \\ 0 & 200 \end{bmatrix} \rightarrow M^{el}_{ij} = \frac{1}{\cancel{100} \cdot 200}\begin{bmatrix} 2\cancel{00} & 0 \\ 0 & 1\cancel{00} \end{bmatrix} = \frac{1}{200}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

$$\bar{M}^{el}_{ij} = \langle M^{el}_{ij} \rangle = \begin{bmatrix} \frac{1}{200} & 0 \\ 0 & 1/100 \end{bmatrix}\frac{1}{3} + \frac{1}{400}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}\frac{1}{3} + \frac{1}{200}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\frac{1}{3} = \begin{bmatrix} \frac{2+3+4}{400} & \frac{-1}{400} \\ \frac{-1}{400} & \frac{3+2}{400} \end{bmatrix}\frac{1}{3}$$

$$= \frac{1}{3 \cdot 400}\begin{bmatrix} 9 & -1 \\ -1 & 5 \end{bmatrix} = \frac{1}{100}\begin{bmatrix} 9/12 & -1/12 \\ -1/12 & 5/12 \end{bmatrix}$$

Anisotropic members



$w^1 = \frac{1}{3}$, $w^2 = \frac{1}{3}$, $w^3 = \frac{1}{3}$ — ∗Each member has the same weights.

member 1, 2, 3
$\theta = 0°, 45°, 90°$
$w = \frac{1}{3}, \frac{1}{3}, \frac{1}{3}$

$$C^{el}_{ij} = \begin{bmatrix} 200 & 0 \\ 0 & 100 \end{bmatrix} [GPa]$$

Note transformation Rule.

$$\sigma'_i = R_{ij}\sigma_j$$
$$\varepsilon^{el'}_i = R_{ij}\varepsilon^{el}_j$$
$$C^{el'}_{ij} = R_{ik}R_{jl}C^{el}_{kl}$$
$$M^{el'}_{ij} = R_{ik}R_{jl}C^{el}_{kl}$$

Now, unlike other previous analyses, the two extreme cases studied led to two distinctive results.

$$\boxed{\begin{cases} \bar{M}^{el} = \frac{1}{100}\begin{bmatrix} 6.75 & -0.75 \\ -0.75 & 6.75 \end{bmatrix} & \text{case 1} \\ \bar{M}^{el} = \frac{1}{100}\begin{bmatrix} 9/12 & -1/12 \\ -1/12 & 5/12 \end{bmatrix} & \text{case 2} \end{cases}}$$

Both in $[GPa^{-1}]$

If the aggregate is loaded as much as $\bar{\sigma} = \begin{bmatrix} 100 \text{ MPa} \\ 200 \text{ MPa} \end{bmatrix}$ what is $\bar{\varepsilon}^{el} = \begin{bmatrix} ? \\ ? \end{bmatrix}$

case 1

$$\underbrace{\frac{1}{100}\begin{bmatrix} 6.75 & -0.75 \\ -0.75 & 6.75 \end{bmatrix}}_{GPa^{-1}}\underbrace{\begin{bmatrix} 100 \\ 200 \end{bmatrix}}_{MPa} = \begin{bmatrix} 6.75 & -0.75 \\ -0.75 & 6.75 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\frac{1}{10} = \begin{bmatrix} 6.75 - 1.5 \\ -0.75 + 13.5 \end{bmatrix}\frac{1}{10} = \begin{bmatrix} 5.25 \\ 12.75 \end{bmatrix}[MPa/GPa]$$

case 2

$$\frac{1}{100}\begin{bmatrix} 9/12 & -1/12 \\ -1/12 & 5/12 \end{bmatrix}\begin{bmatrix} 100 \\ 200 \end{bmatrix} = \begin{bmatrix} 9/12 & -1/12 \\ -1/12 & 5/12 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 9/12 - \frac{2}{12} \\ -1/12 + 5/12 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}\frac{1}{12} = \begin{bmatrix} 5.83\ldots \\ 14.1\ldots \end{bmatrix}[MPa/GPa]$$

Summary.

* Case 1 } led to two different results of $\bar{\varepsilon}^{el}$
* case 2

$$\left\langle (C^{el})^{-1} \right\rangle \neq \left(\langle C^{el} \rangle \right)^{-1}$$

The Question now is, then, what is the correct way of estimating $\bar{M}^{el} \left(= (\bar{C}^{el})^{-1} \right)$

Mathematically, if we can "FIND" a way to estimate $\bar{M}^{el}$ by M^{el}, that means...

$\rightarrow$ $\bar{M}^{el}$ as a function of $(M^{el,1}, M^{el,2} \dots M^{el,n})$

$\rightarrow$ $\bar{M}^{el}$ as a function of $(M^{el}, \boxed{R^1, R^2, R^3 \dots R^n \atop \omega^1, \omega^2, \omega^3 \dots \omega^n})$

crystallographic texture

$\rightarrow$ If we know such a function, we would be able to make use of its "INVERSE".

Summary.

→ when grain property (here we focus $\mathbb{M}^{el}$ $= (\mathbb{C}^{el})^{-1}$)
is "anisotropic" and the grains are not in the same orientation, generally,

$$\langle \mathbb{M}^{el} \rangle \neq \langle (\mathbb{C}^{el})^{-1} \rangle$$

→ we need "better" estimation of overall property of polycrystal

→ In what follows, we study a particular method widely studied, used in literature.