INTRODUCTION TO COMPUTATIONAL PLASTICITY USING FORTRAN

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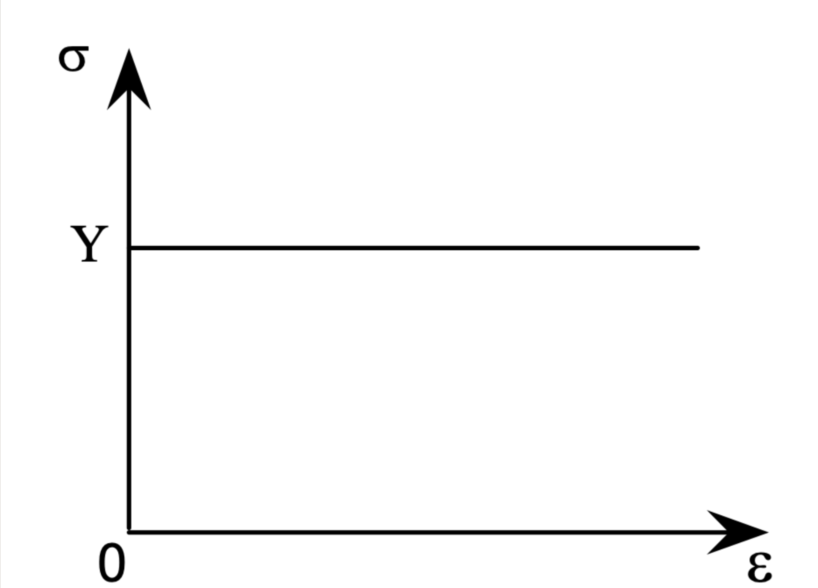
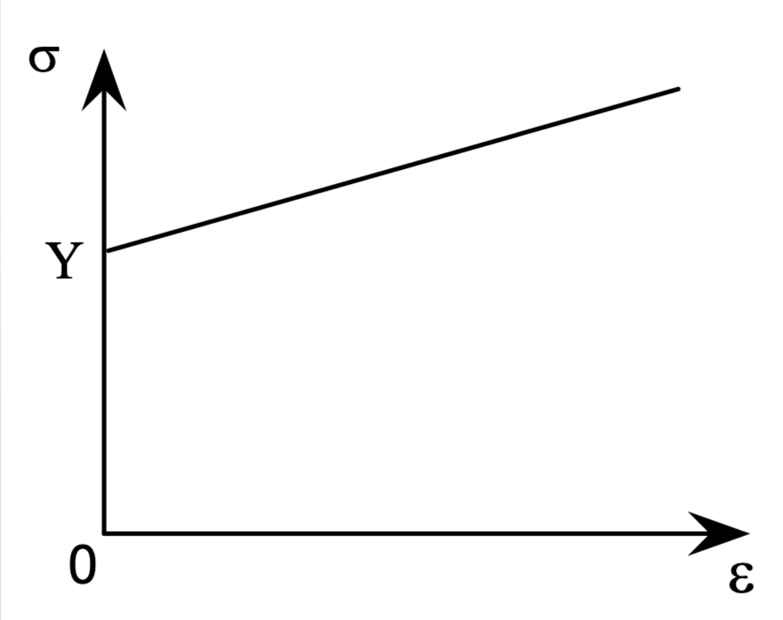
ELASTO-PLASTICITY

- Material that has both elasticity and plasticity

PLASTICITY

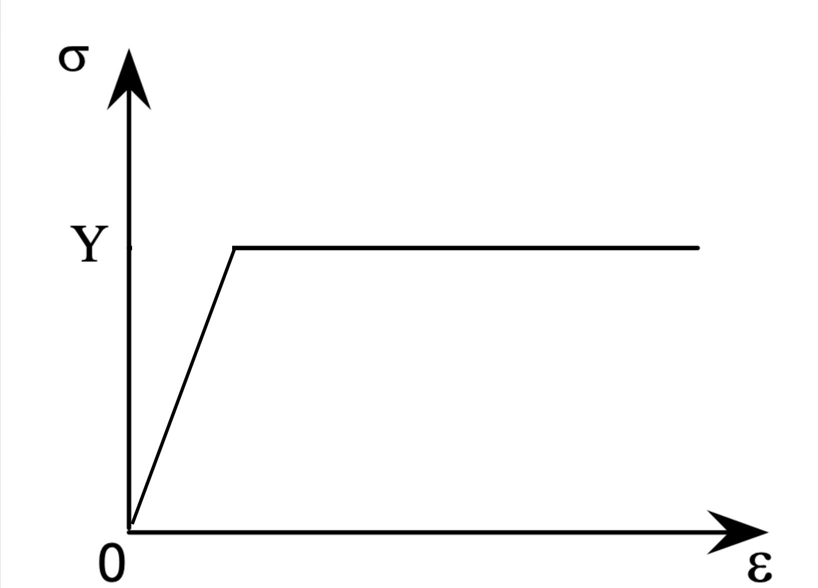


No strain hardening

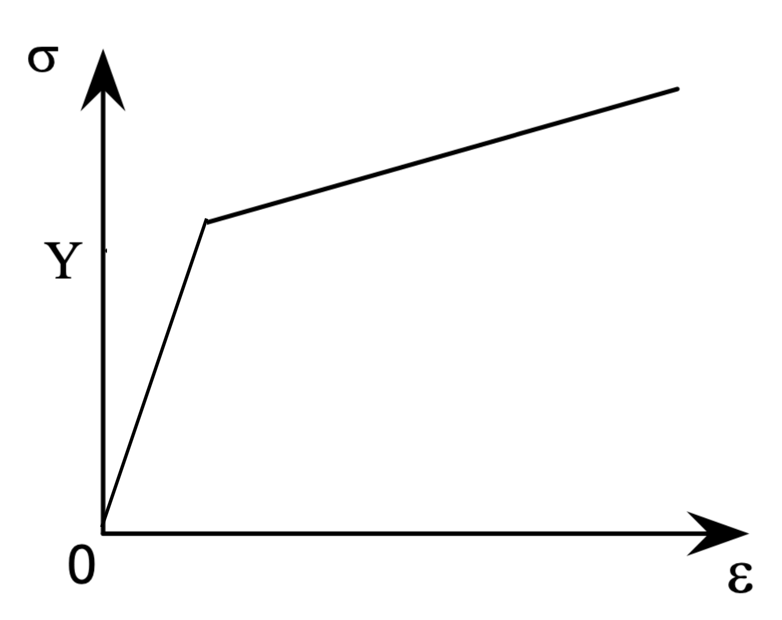


Linear strain hardening

ELASTO-PLASTICITY



No strain hardening



Linear strain hardening

- Elastic constitutive law:
  - $\varepsilon^{el} = \frac{1}{\mathbb{E}}\sigma$
- Plasticity constitutive law:
  - $d\varepsilon^{pl} = \mathbb{H} d\sigma$
- Elasto-plasticity:
  - $d\varepsilon^{pl} + d\varepsilon^{el} = d\varepsilon = \frac{dl}{l}$
  - $d\sigma = \mathbb{E} d\varepsilon^{el} \rightarrow d\sigma = \mathbb{E}\left(d\varepsilon - d\varepsilon^{pl}\right)$

A YIELD CRITERION, NEUTRAL STRESS INCREMENT

Let's say our yield criterion is given as below:

$$f(\sigma) = c$$

A general neutral change $\mathrm{d}\sigma$ (if not yielding any plastic strain) should satisfy below:

$$df = \frac{\partial f}{\partial \sigma} d\sigma = 0$$

Since it remains on the yield locus, no plastic strain is obtained.

Naturally, from the above, we can postulate that $d\varepsilon^{pl} = 0$ for such neutral change of stress. That way, we could speculate something like below is a valid assumption:

$$d\varepsilon^{pl} = Gdf$$

If stress remains on the yield locus (neutral stress increment), no plastic strain is obtained. Then, the next question is what is required for G to be relevant?

(R. Hill, Theory of Plasticity)

G & LAMBDA

- *A most general form of G* would be something like below:

$$G = \lambda \frac{\partial g}{\partial \sigma}$$

with a scalar function g of stress, thus $g \equiv g(\sigma)$

- You do not have to worry about what exactly g is – it is a product of theoretical consideration so far.

ASSOCIATED RULE

- *Arguably, a most interesting and significant assumption that is often made in the theory of metal plasticity is:*

$$g = f(\sigma)$$

- *Since we don't know what g is, why not assuming g is equivalent with yield function f?*
- *Let's see where the above assumption will lead us to..*

$$d\varepsilon^{pl} = Gdf = \lambda \frac{\partial g}{\partial \sigma} df = \lambda \frac{\partial f}{\partial \sigma} df$$

- *With the virtual work principle and the method of Lagrange (not discussed herein), we obtain*

$$d\varepsilon^{pl} = d\lambda \frac{\partial f}{\partial \sigma}$$

We'll find what $d\lambda$ is later

TO BE YIELDED OR NOT-YIELDED THAT IS THE QUESTION

- The condition under which plastic strain increment is non-zero: $f(\sigma) = c$ (yield criterion)
- That means $df \geq 0$
- All the other case: $df < 0$ (pure elastic behavior $d\varepsilon^{pl} = 0$)

ELASTO-RIGID-PLASTICITY

$$\mathrm{d}\varepsilon = d\varepsilon^{el} + d\varepsilon^{pl}$$

Or

$$d\varepsilon = d\varepsilon^{el}$$

Strain as a function of stress

$$\mathrm{d}\varepsilon = \mathbb{E}d\sigma + d\lambda\frac{\partial f}{\partial\sigma}$$

Or

$$d\varepsilon = \mathbb{E}d\sigma$$

$$dw^{pl} = \sigma d\varepsilon^{pl} = \sigma d\lambda\frac{\partial f}{\partial\sigma}$$

Plastic work increment

$$\bar{\sigma}d\bar{\varepsilon}^{pl} = dw^{pl} = \sigma d\varepsilon^{pl} = \sigma d\lambda\frac{\partial f}{\partial\sigma}$$

$$\frac{\bar{\sigma}d\bar{\varepsilon}^{pl}}{\frac{\partial f}{\partial\sigma}\sigma} = d\lambda$$

If we use a homogeneous yield function of degree of 1:

$$\sigma\frac{\partial f(\sigma)}{\partial\sigma} = f(\sigma)$$

$$\frac{\bar{\sigma}d\bar{\varepsilon}^{pl}}{f(\sigma)} = d\lambda$$

If $\mathrm{f}(\sigma) = \bar{\sigma}$,

$d\bar{\varepsilon}^{pl} = d\lambda$

Plastic-work conjugated equivalent strain

$$d\bar{\varepsilon}^{pl} = \frac{\sigma d\varepsilon^{pl}}{f(\sigma)} = d\lambda$$

ELASTO-RIGID-PLASTICITY

$$\mathrm{d}\varepsilon = \mathbb{E}d\sigma + d\lambda\frac{\partial f}{\partial \sigma} = \mathbb{E}d\sigma + \frac{\sigma d\varepsilon^{pl}}{f(\sigma)}\frac{\partial f}{\partial \sigma}$$

Or

$$d\varepsilon = \mathbb{E}d\sigma$$

$$\mathrm{d}\varepsilon = \mathbb{E}d\sigma + \frac{\sigma d\varepsilon^{pl}}{f(\sigma)}\sigma$$

Or

$$d\varepsilon = \mathbb{E}d\sigma$$

$Let's\ say, your\ yield\ function\ f(\sigma)$ is

$$f(\sigma) = \sqrt{\sigma^2}$$

$Your\ yield\ criterion$:

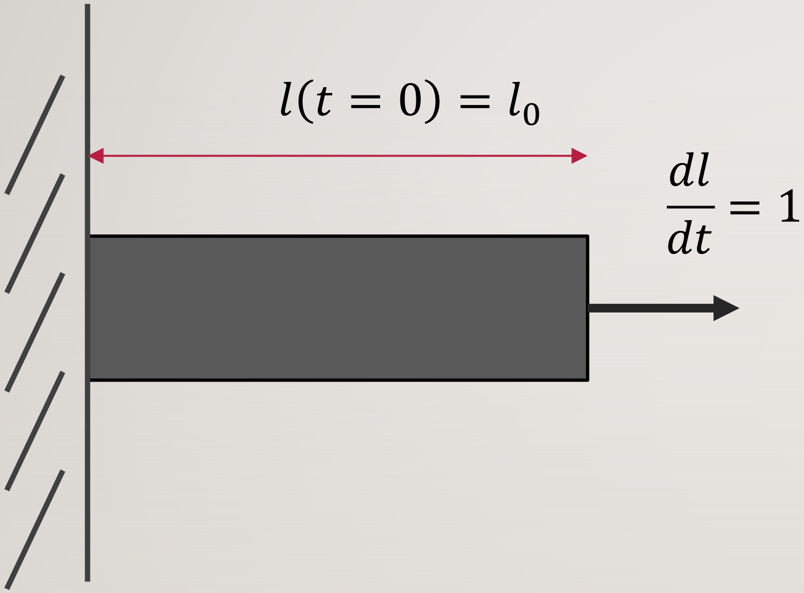
$$f(\sigma) = \sqrt{\sigma^2} = c$$

Say, your material yield property, c, is given as 1.

Your yield function gives

$$\frac{df(\sigma)}{d\sigma} = 1$$

PROBLEM: STRETCHING A ROD



$$Let's\ say, your\ yield\ function\ f(\sigma) \text{ is}$$
$$f(\sigma) = \sqrt{\sigma^2}$$

$$Your\ yield\ criterion:$$
$$f(\sigma) = \sqrt{\sigma^2} = c$$

Say, your material yield property, c, is given as 1.

Your yield function gives

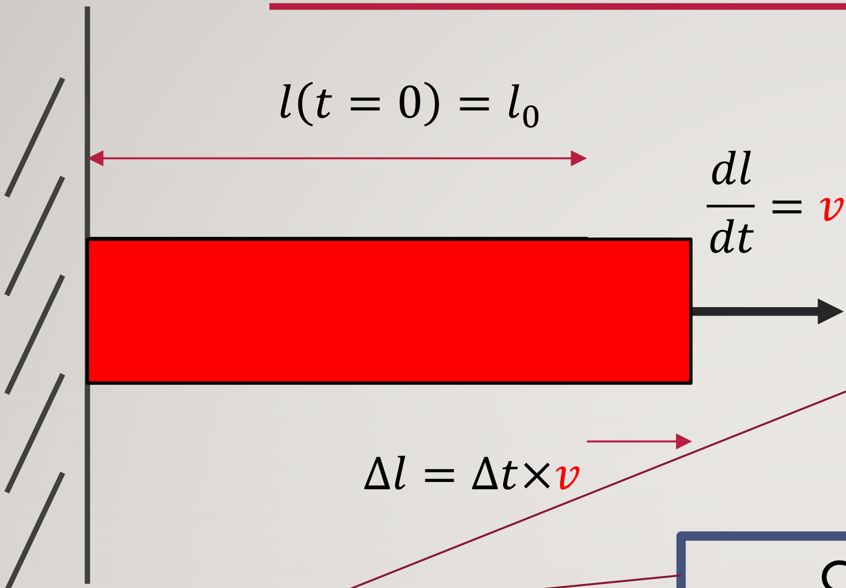
$$\frac{df(\sigma)}{d\sigma} = 1$$

$$\mathrm{d}\varepsilon = \mathbb{E}d\sigma + \frac{\sigma d\varepsilon^{pl}}{f(\sigma)}\sigma$$

Or

$$d\varepsilon = \mathbb{E}d\sigma$$

ELASTIC PREDICTOR AND CORRETOR ALGORITHM



$l(t=0) = l_0$

$$\frac{dl}{dt} = v$$

$$\Delta\varepsilon = \frac{\Delta l}{l}$$

$\Delta l = \Delta t \times v$

Initially assuming that all strain is elastic

$$\Delta\sigma = \mathbb{E}\Delta\varepsilon = \mathbb{E}\frac{\Delta l}{l}$$

Correcting elastic strain $\Delta\sigma = \mathbb{E}(\Delta\varepsilon - \Delta\varepsilon^{pl})$

Stress is updated:

$$\sigma_{(n+1)} = \sigma_{(n)} + \Delta\sigma$$

$$\sigma_{(n+1)} = \sigma_{(n)} + \mathbb{E}\Delta\varepsilon = \sigma_{(n)} + \mathbb{E}\frac{\Delta l}{l}$$

Check if $\sigma_{(n+1)}$ really gives only elastic contribution

$f(\sigma_{(n+1)}) < c$ or $f(\sigma_{(n+1)}) = c$ or $f(\sigma_{(n+1)}) > c$

Plasticity update

$$\Delta\varepsilon^{pl} = \Delta\lambda \frac{\partial f}{\partial \sigma}$$

$$\Delta\varepsilon = \Delta\varepsilon^{el} + \Delta\varepsilon^{pl}$$

You are correct about the strain decomposition, let's move on to next time increment

Elastic predictor and corretor algorithm

$$\Delta l = \Delta t \times v$$

Initially assuming that all strain is elastic

$$\Delta\sigma = \mathbb{E}\Delta\varepsilon = \mathbb{E}\frac{\Delta l}{l}$$

Find the stress corresponding to the current stress increment

$$\sigma_{(n+1)} = \sigma_{(n)} + \Delta\sigma$$

Correcting elastic strain

$$\Delta\varepsilon^{el} = \Delta\varepsilon - \Delta\varepsilon^{pl}$$

That gives new stress increment

$$\Delta\sigma = \mathbb{E}(\Delta\varepsilon - \Delta\varepsilon^{pl})$$

(adjust strain decomposition

Check if $\sigma_{(n+1)}$ is inside, on or over the yield criterion

$f(\sigma_{(n+1)}) < c$ or $f(\sigma_{(n+1)}) = c$ or $f(\sigma_{(n+1)}) > c$

Plastic strain update

$$\Delta\varepsilon^{pl} = \Delta\lambda\frac{\partial f}{\partial\sigma}$$

If $f(\sigma_{(n+1)}) \leq c$

$\sigma_{(n+1)}$ is consistent with our theory.

Let's move on to next time increment

Plastic strain update

$$\Delta\varepsilon^{pl} = \Delta\lambda \frac{\partial f}{\partial \sigma}$$

$F(\Delta\lambda) = f(\Delta\lambda) - c,$ find $\Delta\lambda$ that gives $F(\Delta\lambda) = 0$

That's exactly what NR can do.

$$\frac{dF(\Delta\lambda)}{d\Delta\lambda} = \frac{d(f(\sigma) - c)}{d\Delta\lambda} = \frac{df(\sigma)}{d\Delta\lambda} = \frac{df(\sigma)}{d\sigma}\frac{d\sigma}{d\Delta\lambda} = 1\times\frac{d(\sigma_{(n)} + \Delta\sigma)}{d\Delta\lambda} = \frac{d\Delta\sigma}{d\Delta\lambda}$$

$$= \frac{d\mathbb{E}\left(\Delta\varepsilon - \Delta\varepsilon^{pl}\right)}{d\Delta\lambda} \approx -\mathbb{E}\frac{d\Delta\varepsilon^{pl}}{d\Delta\lambda} = -\mathbb{E}\,\mathrm{sgn}\left(\Delta\varepsilon^{pl}\right)$$

$\Delta\varepsilon^{pl} = \Delta\bar{\varepsilon}^{\mathrm{pl}}$ if $\Delta\varepsilon^{pl} \geq 0$
$\Delta\varepsilon^{pl} = -\Delta\bar{\varepsilon}^{\mathrm{pl}}$ if $\Delta\varepsilon^{pl} < 0$

$$\Delta\lambda_{(k+1)} = \Delta\lambda_{(k)} - \frac{F\left(\lambda_{(k)}\right)}{-\mathbb{E}\,\mathrm{sgn}(\Delta\varepsilon)}$$

Caution! Need validation

CHEAT SHEET

```fortran
      real function calc_yield_function(stress)
      implicit none
      real stress
      calc_yield_function = sqrt(stress**2.)
      return
      end function
```

```fortran
      program elasto_plasticity_scalar
      implicit none
      real calc_yield_function
      real dt, E, c, t, l, dl, eps, deps, deps_el, deps_pl,dsig,
     $     stress
      real vel, f, tol
      integer kount,iplast
      parameter(tol=1e-6)

      open(3,file='elasto_plasticity_scalar.txt')
      dt = 0.02                     ! time increment
      E = 200000                    ! elastic modulus
      c = 200.                      ! yield criterion
      stress = 0.                   ! initial stress
      eps = 0.                      ! initial strain
      l = 1.                        ! initial length
      t = 0.                        ! initial time
      do while(t<1.0)
c     Loading condition 1
         if (t.le.0.25) then
            vel = 0.01
         elseif (t.gt.0.25.and.t.le.0.55) then
            vel =-0.01
         else
            vel = 0.01
         endif
         dl = vel * dt
         deps = dl / l
c        initially assuming all strain is elastic
         deps_pl = 0.0
         deps_el = deps
c        guess on stress increment
         dsig = E * deps_el
         kount = 0
         f = calc_yield_function( stress+ dsig) - c
         iplast=0
         do while (f.gt.tol.and.kount.lt.3) ! if exceeding plastic onset
            iplast=1
            deps_el = deps - deps_pl
            dsig = e * deps_el
            f = calc_yield_function(stress+dsig) - c
c           estimate new plastic increment
            deps_pl = deps_pl - f/(-E)*sign(1.,deps)
            kount = kount +1
         enddo
         write(3,'(2f9.4,2f10.4,2i2)')t,l,eps,stress,iplast,kount
         stress = stress + dsig
         eps = eps + deps
         t = t + dt
         l = l + dl
      enddo
      close(3)
      end program
```

IN THE CASE OF HARDENING MATERIAL

- Let's say our material obeys the below strain-hardening rule:
  - $\bar{\sigma} = \bar{\sigma}_0 + K\left(\bar{\varepsilon}^{pl}\right)^n$
- And, our yield surface isotropically expands (strain hardening)
- Our yield condition (not yield function) changes to
  - $f(\sigma) = \bar{\sigma} = \bar{\sigma}_0 + K\left(\bar{\varepsilon}^{pl}\right)^n$
- The previously constant replaced by the equivalent stress that represents the size of yield surface (hardening).

Plastic strain update

$$\Delta\varepsilon^{pl} = \Delta\lambda \frac{\partial f}{\partial \sigma}$$

$F(\Delta\lambda) = f(\Delta\lambda) - \left\{\bar{\sigma}_0 + K\left(\bar{\varepsilon}^{pl}\right)^n\right\}$, find $\Delta\lambda$ that gives $F(\Delta\lambda) = 0$

That's exactly what NR can do.

$$\frac{dF(\Delta\lambda)}{d\Delta\lambda} = \frac{d\left(f(\sigma) - \bar{\sigma}_0 - K\left(\bar{\varepsilon}^{pl}\right)^n\right)}{d\Delta\lambda} = \frac{df(\sigma)}{d\Delta\lambda} - \frac{d\left\{\bar{\sigma}_0 + K\left(\bar{\varepsilon}^{\mathrm{pl}}_{(n)} + \Delta\bar{\varepsilon}^{\mathrm{pl}}\right)^n\right\}}{d\Delta\lambda}$$

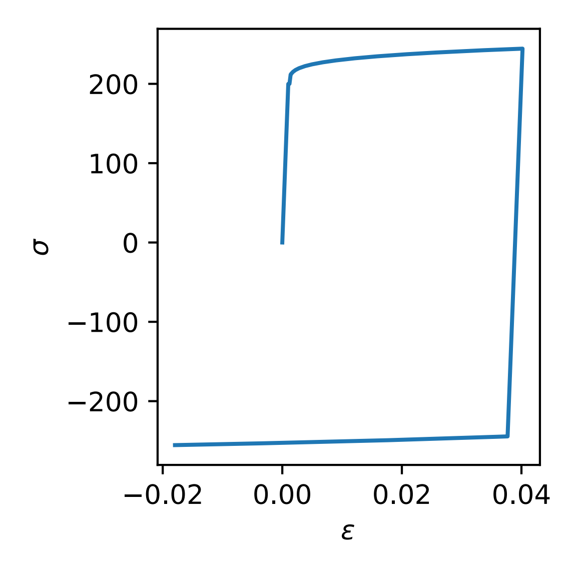
$$= \frac{df(\sigma)}{d\sigma}\frac{d\sigma}{d\Delta\lambda} - K\frac{d\left(\bar{\varepsilon}^{\mathrm{pl}}_{(n)} + \Delta\bar{\varepsilon}^{\mathrm{pl}}\right)^n}{d\Delta\bar{\varepsilon}^{\mathrm{pl}}}\frac{d\Delta\bar{\varepsilon}^{pl}}{d\Delta\lambda}$$

$$= \frac{d\mathbb{E}\left(\Delta\varepsilon - \Delta\varepsilon^{pl}\right)}{d\Delta\lambda} - nK\left(\bar{\varepsilon}^{\mathrm{pl}}_{(n)} + \Delta\bar{\varepsilon}^{\mathrm{pl}}\right)^{n-1}$$

$$\Delta\lambda_{(k+1)} = \Delta\lambda_{(k)} - \frac{F\left(\Delta\lambda_{(k)}\right)}{-\mathbb{E}\,\mathrm{sgn}(\Delta\varepsilon) - nK\left(\bar{\varepsilon}^{\mathrm{pl}}_{(n)} + \Delta\bar{\varepsilon}^{\mathrm{pl}}\right)^{n-1}}$$

Caution! Need validation

A TESTED CASE



Material properties:
Hollomon Eq.

$$\bar{\sigma} = \bar{\sigma}_0 + K\left(\bar{\varepsilon}^{pl}\right)^n$$

With

$$\bar{\sigma}_0 = 200$$
$$K = 100$$
$$n = 0.25$$
$$\mathbb{E} = 200{,}000$$

Tension/Compression