Introduction to computational plasticity using FORTRAN

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Build a sub program on von Mises

- Main program accepts stress tensor
- FUNCTION that represents the von Mises equivalent stress based on the given stress tensor

$$\sqrt{\frac{(\sigma^{I}-\sigma^{II})^2+(\sigma^{II}-\sigma^{III})^2+(\sigma^{I}-\sigma^{III})^2}{2}}=\sigma^{vm}$$

von Mises function

```
real function vonmises(stress)
real stress(3)

vonmises=(stress(1)-stress(2))**2+(stress(2)-stress(3))**2+(stress(3)-stress(1))**2
vonmises=sqrt(vonmises/2.)

return
end function
```

Build a program called von Mises

- Now with full Cauchy stress tensor

$$\sqrt{\frac{(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2+6(\sigma_{12}^2+\sigma_{23}^2+\sigma_{13}^2)}{2}} = \phi^{vm}$$

Build a program called von Mises

- Now, let’s write a **subroutine** version von Mises function.

von mises subroutine

```
subroutine vonmises(stress,vm)
real stress(3), vm

vm=(stress(1)-stress(2))**2+(stress(2)-
stress(3))**2+(stress(3)-stress(1))**2
vm=sqrt(vm/2.)

return
end subroutine
```

```
program main
real stress(3)
stress(1)=1.
stress(2)=0.
stress(3)=1.
call vonmises(stress,vm)
write(*,*)'von Mises stress:',vm
end
```

Find stress that satisfies yield criterion

- In the plane of $(\sigma_{11}, \sigma_{22})$ assuming that the other stress components are zero.
- We take an approach utilizing the characteristics of 'homogeneous function of degree n'

- This is not the only way; there can be numerous ways to find stresses that satisfies yield criterion. But the method we are going to use is quite useful to understand the mathematical characteristic of homogeneous yield function.

Metal flow theory with homogeneous function

Then, what is a homogeneous function? What properties should we know?

A homogeneous function (*f*) of degree *n* in the space of (x,y) obeys:

$f(tx, ty) = t^n f(x, y)$ where *t* is any arbitrary constant

Function *f* is called as a homogeneous function of degree *n*

To make sure your stress coordinates on the yield locus (surface)

1. Choose an arbitrary stress in the space **t$\boldsymbol{\sigma}$** (individual unknown, yet, the product is specified arbitrarily by yourself). Note that **t$\boldsymbol{\sigma}$** can be considered as a vector in the arbitrary stress space.

2. Evaluate the result

$\tilde{\phi} = f(\mathrm{t}\boldsymbol{\sigma})$; since t$\boldsymbol{\sigma}$ is specified, $\tilde{\phi}$ is also automatically specified as well.

3. Using the characteristic of homogeneous function of degree n, the above means:

$$\tilde{\phi}t^n = f(\boldsymbol{\sigma})$$

4. Now, the usefulness of above becomes obvious: take a look at below chain of rearrangements

Our preliminary result is summarized to be: $f(t\boldsymbol{\sigma}) = \tilde{\phi}$

Next question will be: what is r that satisfies $f(r\boldsymbol{\sigma}) = 1$?

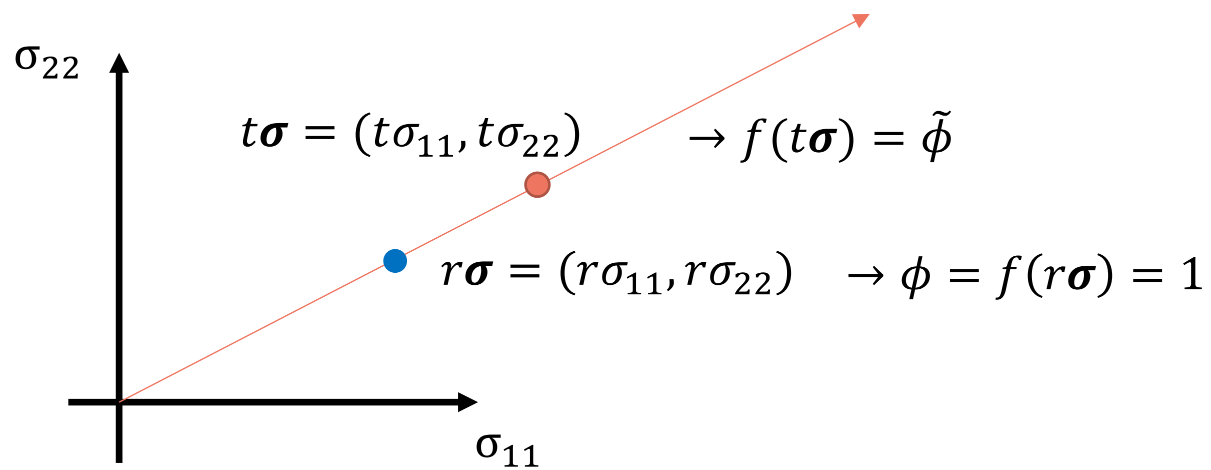
$$\frac{1}{\tilde{\phi}} = \frac{f(r\boldsymbol{\sigma})}{f(t\boldsymbol{\sigma})} = \frac{r^n f(\boldsymbol{\sigma})}{t^n f(\boldsymbol{\sigma})} = \left(\frac{r}{t}\right)^n \quad \rightarrow \frac{r}{t} = \left(\frac{1}{\tilde{\phi}}\right)^{1/n} \rightarrow r = t\left(\frac{1}{\tilde{\phi}}\right)^{1/n}$$

$$\rightarrow r\boldsymbol{\sigma} = t\boldsymbol{\sigma}\left(\frac{1}{\tilde{\phi}}\right)^{1/n}$$

Now that we obtain $r\boldsymbol{\sigma}$, we know $r\boldsymbol{\sigma}$ is located on top of yield locus with $f(r\boldsymbol{\sigma}) = 1$.

Scaling a ray of stress to meet yield surface

A coordinate (σ_x, σ_y) can be regarded as a vector $\boldsymbol{\sigma}$ with its starting point fixed to origin (0,0)
Having the above in mind, let's look at the the scaling …



Algorithm to find where the stress ray meets the yield surface

- Goal: find a point vector $\mathbf{r}\boldsymbol{\sigma}$ with a specified size $|\mathbf{r}\boldsymbol{\sigma}| = 1$ for a homogeneous function of degree n
- Algorithm
  - Obtain $\tilde{\phi} = f(\mathrm{t}\boldsymbol{\sigma})$
  - Obtain $r\boldsymbol{\sigma} = t\boldsymbol{\sigma}\left(\frac{1}{\tilde{\phi}}\right)^{1/n}$
  - $r\boldsymbol{\sigma}$ now gives $\phi(r\sigma) = 1$

Example

```
subroutine vonmises(stress,vm)
real stress(3), vm

vm=(stress(1)-stress(2))**2+(stress(2)-stress(3))**2+(stress(3)-stress(1))**2
vm=sqrt(vm/2.)

return
end subroutine
```

- Algorithm
  - Obtain $\tilde{\phi} = f(\mathbf{t\sigma})$
  - Obtain $r\boldsymbol{\sigma} = t\boldsymbol{\sigma}\left(\frac{1}{\tilde{\phi}}\right)^{1/n}$
  - $r\boldsymbol{\sigma}$ now gives $\phi(r\sigma) = 1$

```
subroutine yf(tsig,rsig)
real tsig(3),rsig(3),phitilde,phi
Integer n
n=1 !! homogeneous
call vonmises(tsig,phitilde)

do i=1, 3
  rsig(i)=tsig(i) * (1./phitilde)**(1./n)
enddo

call vonmises(rsig,phi)
Write(*,*)'phi:',phi
return
end subroutine
```

```
program yf(stress)
Real tsig(3)
tsig(1)=1./sqrt(3.)
tsig(2)=1./sqrt(3.)
tsig(3)=0.

call yf(tsig,rsig)

write(*,*)'rsig:',rsig(1),rsig(2),rsig(3)

end program
```

Hill yield function (Exercise)

- Add another choice of yield function: Hill48
- In below form, the Hill48 is a homogeneous yield function of degree n with respect to stress tensor

$$\phi^{\mathrm{H48}} = \{F(\sigma_{22}-\sigma_{33})^2 + G(\sigma_{33}-\sigma_{11})^2 + H(\sigma_{11}-\sigma_{22})^2 + 2L\sigma_{23}^2 + 2M\sigma_{31}^2 + 2N\sigma_{12}^2\}^{\frac{1}{2}}$$

Hill48 yield function (cheat sheet)

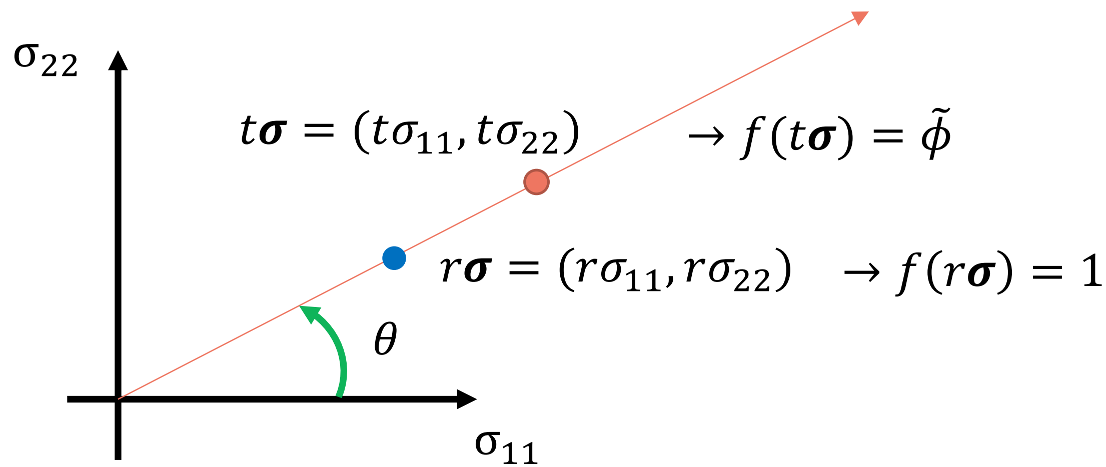
```
subroutine hill48(f,g,h,l,m,n,s,hill)
implicit none
real f,g,h,l,m,n
real s(6),hill
real s1,s2,s3,s4,s5,s6
s1=s(1)
s2=s(2)
s3=s(3)
s4=s(4)
s5=s(5)
s6=s(6)
hill=f*(s2-s3)**2+g*(s3-s1)**2+h*(s1-s2)**2 + 2*l*s4**2+2*m*s5**2
$     +2*n*s6**2
hill = sqrt(hill)

write(*,*)'hill:',hill
write(*,'(6f10.2)')s1,s2,s3,s4,s5,s6
return
end subroutine hill48
```

Improve von Mises program

- Add a feature to calculate derivative
  - Analytical derivative
  - Numerical derivative (based on finite difference)

$$\sqrt{\frac{(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2+6(\sigma_{12}^2+\sigma_{23}^2+\sigma_{13}^2)}{2}}=\phi^{vm}$$

Hill, von Mises yield surfaces



Find various points of $r\boldsymbol{\sigma}$ within

$$-\pi \leq \theta \leq +\pi$$

That gives yield locus with $\phi = 1$

Hill, von Mises yield surfaces (cheat sheet)

```fortran
      program yf_plot
      implicit none
      real tsig(6),rsig(6),dangle,pi,angle,ang,rsig2(6)
      integer nangle
      parameter(nangle=360)
      integer i,j

      pi=3.141592
      dangle=2.*pi/(nangle-1)
      open(1,file='ys.txt',status='unknown')
      do i=1,nangle
         ang=-pi + dangle*(i-1.)
         tsig(1)=cos(ang)
         tsig(2)=sin(ang)
         tsig(3:)=0.

         call yf(tsig,rsig2,1)   ! von Mises
         call yf(tsig,rsig,2)   ! Hill 48
         write(1,'(9f10.4)')(tsig(j),j=1,3),(rsig(j),j=1,3),
     $        (rsig2(j),j=1,3)
      enddo
      close(1)
      end program
```

-UUU:----F1 yf_plot.f   All L17   (Fortran) [yf_plot] ---

```fortran
      subroutine hill48(f,g,h,l,m,n,s,hill)
      implicit none
      real f,g,h,l,m,n
      real s(6),hill
      real s1,s2,s3,s4,s5,s6
      s1=s(1)
      s2=s(2)
      s3=s(3)
      s4=s(4)
      s5=s(5)
      s6=s(6)
      hill=f*(s2-s3)**2+g*(s3-s1)**2+h*(s1-s2)**2 + 2*l*s4**2+2*m*s5**2
     $     +2*n*s6**2
      hill = sqrt(hill)

      write(*,*)'hill:',hill
      write(*,'(6f10.2)')s1,s2,s3,s4,s5,s6
      return
      end subroutine hill48
```

-UUU:----F1 hill.f   All L17   (Fortran) [hill48] ---

```fortran
      subroutine vonmises(s,vm)
      implicit none
      real s(6),vm
      real s1,s2,s3,s4,s5,s6

      s1=s(1)
      s2=s(2)
      s3=s(3)
      s4=s(4)
      s5=s(5)
      s6=s(6)

      vm = (s1-s2)**2+(s2-s3)**2+(s3-s1)**2 + 6*(s4**2+s5**2+s6**2)
      vm = sqrt(vm /2.)
      write(*,*)'vm:',vm
      return
      end subroutine
```

```fortran
      subroutine yf(tsig,rsig,iopt)
      real tsig(6),rsig(6),phitilde,phi
      integer n,iopt
      !! Von Mises
      if (iopt.eq.1) then
         n=1.
         call vonmises(tsig,phitilde)
         write(*,*)'phitilde:',phitilde
         do i=1,6
            rsig(i)=tsig(i)*(1./phitilde)**(1./n)
         enddo
         write(*,*)(rsig(i),i=1,3)
         call vonmises(rsig,phi)
         write(*,*)'phi (Von Mises):',phi
      elseif(iopt.eq.2) then
         n=1.
         call hill48(0.5,0.5,0.5,0.5,0.5,0.5,tsig,phitilde)
         do i=1,6
            rsig(i)=tsig(i)*(1./phitilde)**(1./n)
         enddo
         call hill48(0.5,0.5,0.5,0.5,0.5,0.5,rsig,phi)
      endif

      return
      end subroutine yf
```

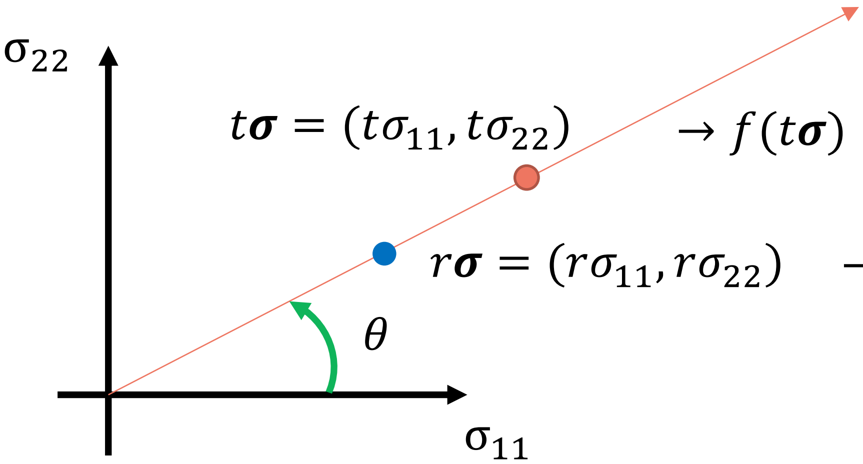
Comparing von Mises and Hill 48

- Hill 48 with F,G,H=0.5,L,M,N=1.5 reduces to von Mises

$$\phi^{\mathrm{H48}} = \left\{\frac{1}{2}(\sigma_{22}-\sigma_{33})^2 + \frac{1}{2}(\sigma_{33}-\sigma_{11})^2 + \frac{1}{2}(\sigma_{11}-\sigma_{22})^2 + \sigma_{23}^2 + \sigma_{31}^2 + \sigma_{12}^2\right\}^{\frac{1}{2}}$$

$$\sqrt{\frac{(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2)}{2}} = \phi^{\mathrm{vm}}$$

Hill, von Mises yield surfaces with arbitrary size of yield surface



$t\boldsymbol{\sigma} = (t\sigma_{11}, t\sigma_{22})$ $\rightarrow f(t\boldsymbol{\sigma}) = \tilde{\phi}$

$r\boldsymbol{\sigma} = (r\sigma_{11}, r\sigma_{22})$ $\rightarrow \phi = f(r\boldsymbol{\sigma}) = 1$

θ

Find various points of $r\boldsymbol{\sigma}$ within

$$-\pi \leq \theta \leq +\pi$$

That gives yield locus with $\phi = 1$

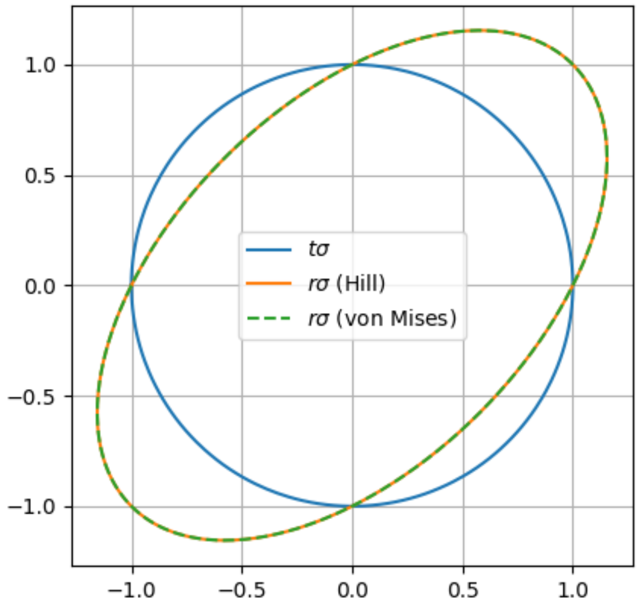
Can you obtain stress coordinates that satisfy a new yield surface of $\phi^{new} = 3$?

From $\phi = f(r\boldsymbol{\sigma}) = 1$

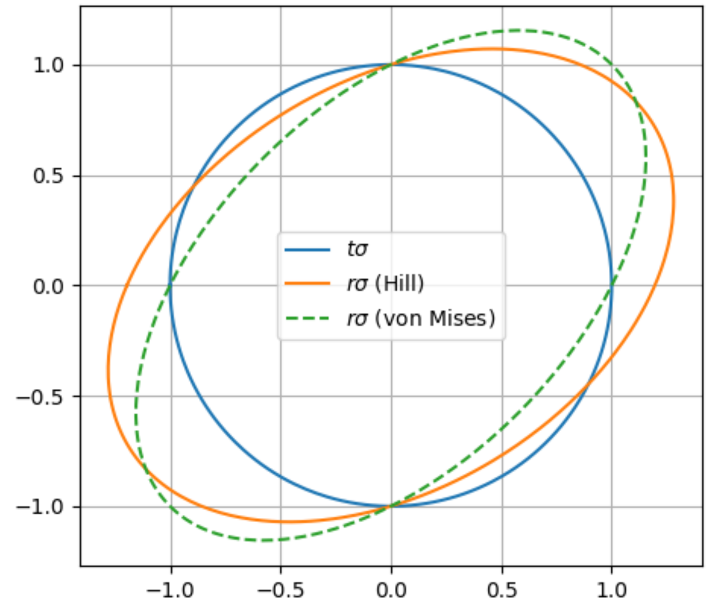
$$\phi^{new} = \phi \times 3 = 3f(r\boldsymbol{\sigma}) = f(3 \times r\boldsymbol{\sigma})$$

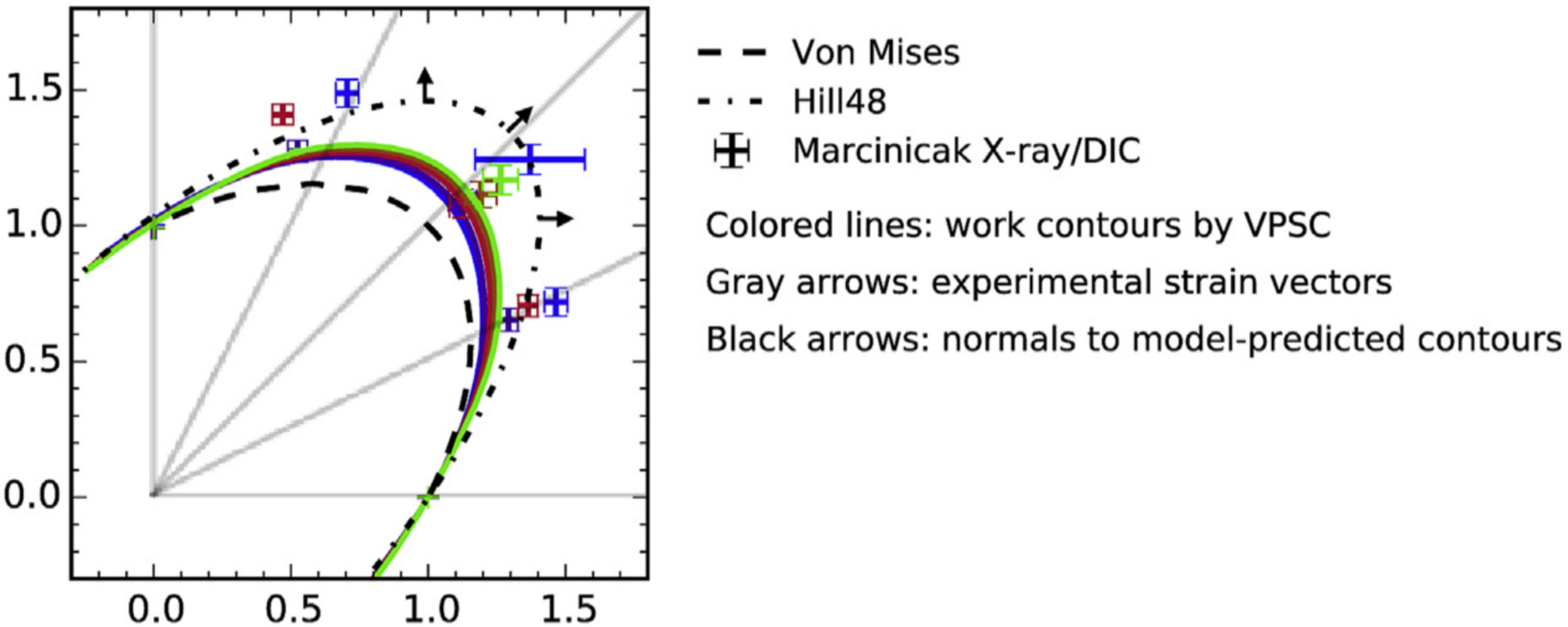
Comparing von Mises and Hill 48

F=0.5, G=0.5, H=0.5,
L=M=N=1.5



F=0.7, G=0.4, H=0.3,
L=M=N=1.5





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Derivative of yield function (analytical solution)

- Derivatives of yield function plays an important role in theory of plasticity

$$\frac{\partial \phi^{vm}}{\partial \sigma_{ij}}?$$

$$\sqrt{\frac{(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2+6(\sigma_{12}^2+\sigma_{23}^2+\sigma_{13}^2)}{2}}=\phi^{vm}$$

Analytical solution can be estimated by taking derivatives by yourself.

EX.

$$\frac{\partial \phi^{vm}}{\partial \sigma_{11}}=\frac{1}{2}(\phi^{vm})^{-\frac{1}{2}}\times\frac{1}{2}(2\sigma_{11}-2\sigma_{22}+2\sigma_{11}-2\sigma_{33})$$

Derivative of yield function (finite difference)

- Definition of partial derivatives

$$\frac{\partial \phi^{\mathrm{vm}}}{\partial \sigma_{ij}} = \lim_{\Delta\sigma_{ij}\to 0} \frac{\phi^{vm}\left(\sigma_{ij} + \Delta\sigma_{ij}\right) - \phi^{vm}\left(\sigma_{ij}\right)}{\Delta\sigma_{ij}}$$

This above can be approximated by finite difference when the difference in the relevant stress component is sufficiently small.

$$\frac{\partial \phi^{\mathrm{vm}}}{\partial \sigma_{ij}} \approx \frac{\phi^{vm}(\sigma_{ij}+\Delta\sigma_{ij})-\phi^{vm}(\sigma_{ij})}{\Delta\sigma_{ij}}$$

Provided that $\Delta\sigma_{ij}$ is sufficiently small

Numerical derivative of yield function (cheat sheet)

```
c     program to numerically obtain derivatives of yield surface

      program ys_diff
      implicit none
      real s(6),sd(6),rsig(6),s_plus_delta(6),phi,phi1,phi2,delta,
     $     deltas(10)
      integer n,i


      s(:)=0.
      s(1)=1.
      s(2)=1.

      n = 1 ! degree of homogeneous function
      call vonmises(s,phi) ! phi tilde
      do i=1,6
         s(i)=s(i)*(1./phi)**(1./n)
      enddo
c     s is now on the VM yield surface with phi=1.
      call vonmises(s,phi)      ! phi should be one (if I have not screwed up so far)

c     Its derivative?
      delta=0.0001
      do i=1,6 ! loop over each component
         s_plus_delta(:)=s(:)   ! initialize
         s_plus_delta(i)=s_plus_delta(i)+delta ! only for specific component
         call vonmises(s_plus_delta,phi1)
         sd(i) = (phi1-phi)/delta       ! this is the finite difference
      enddo

      write(*,'(6f10.3)') (sd(i),i=1,6)

      end program
```