# 비정상 냉각
# Non-Steady State Cooling

이동현상이론 (MSA0013)


창원대학교 신소재공학부
정영웅

yjeong@changwon.ac.kr
https://youngung.github.io
https://github.com/youngung

# 일반적 열전도 방정식

$$-k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\dot{q}=\rho C_{p(m)}\frac{\partial T}{\partial t}$$

Steady-state

$$-k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\dot{q}=0$$

One dimensional ($x$) non Steady-state
without internal heat generation

$$-k\left(\frac{\partial^2 T}{\partial x^2}\right)=\rho C_{p(m)}\frac{\partial T}{\partial t}$$

$$\frac{\partial T}{\partial t}=\alpha\left(\frac{\partial^2 T}{\partial x^2}\right)$$

$T$는 공간 ($x$) 와 시간 ($t$)의 함수. ODE?

# Solution of Heat equation

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2}\right)$$

$T$는 공간 ($x$) 와 시간 ($t$)의 함수. ODE? (변수가 하나인 미분 방정식)

주로, Fundamental solution을 얻는다. 주어진 경계 조건에 적절한 Green function을 찾는다.

$$T(x,t) = \int \Phi(x-y,t)g(y)dy$$

우리가 사용하는 해결 방식은 무차원 (dimensionless) 단일 변수를 찾는 것에서부터.

$$\eta = \frac{x}{2\sqrt{\alpha t}}$$

$$\frac{\partial \eta}{\partial x} = \frac{1}{2\sqrt{\alpha t}} \qquad \frac{\partial \eta}{\partial t} = -\frac{x}{4\sqrt{\alpha t^3}}$$

$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial \eta}\cdot\frac{\partial \eta}{\partial t} = -\frac{x}{4\sqrt{\alpha t^3}}\left(\frac{\partial T}{\partial \eta}\right)$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial\left[\frac{\partial T}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}\right]}{\partial x} = \frac{\partial\left[\frac{\partial T}{\partial \eta}\cdot\frac{1}{2\sqrt{\alpha t}}\right]}{\partial x}$$

$$= \frac{1}{2\sqrt{\alpha t}}\frac{\partial\left[\frac{\partial T}{\partial \eta}\right]}{\partial x} = \frac{1}{2\sqrt{\alpha t}}\frac{\partial\left[\frac{\partial T}{\partial \eta}\right]}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = \frac{1}{2\sqrt{\alpha t}}\frac{\partial^2 T}{\partial \eta^2}\cdot\frac{1}{2\sqrt{\alpha t}}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{4\alpha t}\frac{\partial^2 T}{\partial \eta^2}$$

$$-\frac{x}{4\sqrt{\alpha t^3}}\left(\frac{\partial T}{\partial \eta}\right) = \frac{\alpha}{4\alpha t}\frac{\partial^2 T}{\partial \eta^2}$$

$$-\frac{x}{\sqrt{\alpha t}}\left(\frac{\partial T}{\partial \eta}\right) = \frac{\partial^2 T}{\partial \eta^2}$$

$$-2\eta\left(\frac{\partial T}{\partial \eta}\right) = \frac{\partial^2 T}{\partial \eta^2}$$

$$\left(\frac{dT}{d\eta}\right) = -\frac{1}{2\eta}\frac{d^2 T}{d\eta^2}$$

Put $y = \frac{dT}{d\eta}$ gives

$$y = -\frac{1}{2\eta}\frac{dy}{d\eta}$$

# Solution of Heat equation

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2}\right)$$

Put $y = \frac{dT}{d\eta}$ gives

$$y = -\frac{1}{2\eta}\frac{dy}{d\eta}$$

$$-2\eta d\eta = \frac{1}{y}dy$$

$$-\eta^2 = \ln y + C$$

Put $C$ as $\ln A$ gives

$$-\eta^2 = \ln y + \ln A$$

$$-\eta^2 = \ln\left(\frac{y}{A}\right)$$

$$\frac{y}{A} = \exp(-\eta^2)$$

$$y = A \cdot \exp(-\eta^2)$$

Remember $y = \frac{dT}{d\eta}$

$$\frac{dT}{d\eta} = A \cdot \exp(-\eta^2)$$

$$dT = A \cdot \exp(-\eta^2)\ d\eta$$

적분하면?

$$\int_{T_0}^{T_1} dT = A\int_{\eta_0}^{\eta_1} \exp(-\eta^2)\ d\eta$$

$\eta = \infty$ (즉, $\eta = \frac{x}{2\sqrt{\alpha t}} = \infty$) 인 조건은 모든 $x$에서 (즉 어느 공간점에서) $t = 0$인 조건과 동일하다.

$\eta = \infty$ 조건 → $t = 0$ 이며 모든 공간에서 동일한 온도

$$\int_{T_0}^{T_i} dT = A\int_{0}^{\infty} \exp(-\eta^2)\ d\eta$$

$\eta = 0$ (즉, $\eta = \frac{x}{2\sqrt{\alpha t}} = 0$) 인 조건은 모든 $t$에서 (즉 어느 시점이든) $x = 0$인 조건과 동일하다.

$\eta = 0$ 조건 → $x = 0$ 이며 어느 시점에서든 동일한 온도

$$T_i - T_0 = A\frac{\sqrt{\pi}}{2}$$

$$\frac{2}{\sqrt{\pi}}(T_i - T_0) = A$$

$$\int_{T}^{T_i} dT = A\int_{\eta}^{\infty} \exp(-\eta^2)\ d\eta$$

$$T_i - T = \frac{2}{\sqrt{\pi}}(T_i - T_0)\int_{\eta}^{\infty} \exp(-\eta^2)\ d\eta$$

$$\frac{T_i - T}{(T_i - T_0)} = \frac{2}{\sqrt{\pi}}\int_{\eta}^{\infty} \exp(-\eta^2)\ d\eta$$

$$\frac{2}{\sqrt{\pi}}\int_{\eta}^{\infty} \exp(-\eta^2)\ d\eta = \frac{2}{\sqrt{\pi}}\int_{0}^{\infty} \exp(-\eta^2)\ d\eta - \frac{2}{\sqrt{\pi}}\int_{0}^{\eta} \exp(-\eta^2)\ d\eta$$

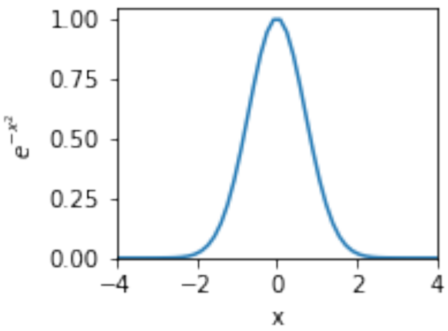
$$\frac{2}{\sqrt{\pi}}\int_{\eta}^{\infty} \exp(-\eta^2)\ d\eta = \frac{2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} - \mathrm{erf}(\eta) = 1 - \mathrm{erf}(\eta)$$

$$\frac{T_i - T}{(T_i - T_0)} = 1 - \mathrm{erf}(\eta)$$

$$\frac{T - T_i}{(T_0 - T_i)} = 1 - \mathrm{erf}(\eta)$$

# Exp (-x^2) and erf

Bell 모양 곡선
$y(x) = e^{-x^2}$




$y(0) = e^{-0^2} = 1$

$x \to \pm\infty, y \to 0$

$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ $\quad \int_{0}^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Error function; erf.

$$\mathrm{erf}(x) = \frac{\int_0^x e^{-x^2} dx}{\int_0^\infty e^{-x^2} dx} = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx$$

$$\mathrm{erf}(x) \approx \frac{2}{\sqrt{\pi}} \frac{3x}{x^2 + 3}$$

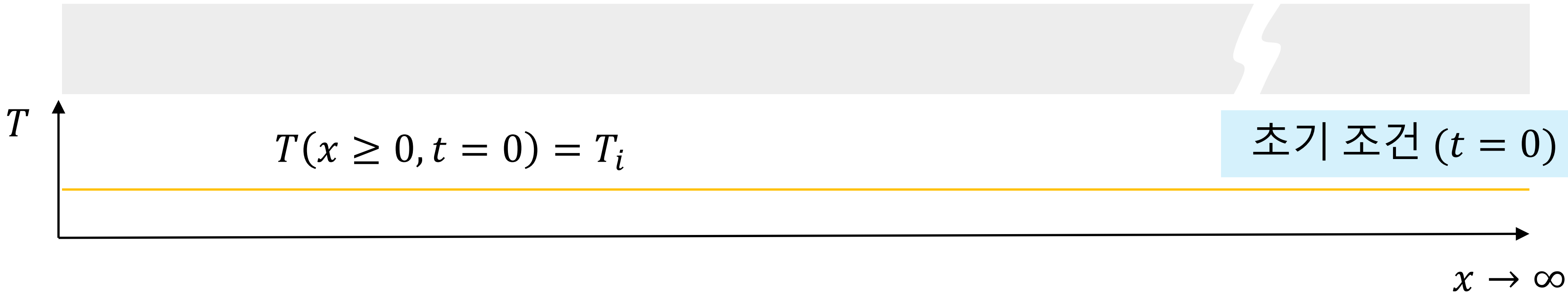
$$\mathrm{erfc}(x) \approx 1 - \mathrm{erf}(x)$$

# 반무한 고체의 전도

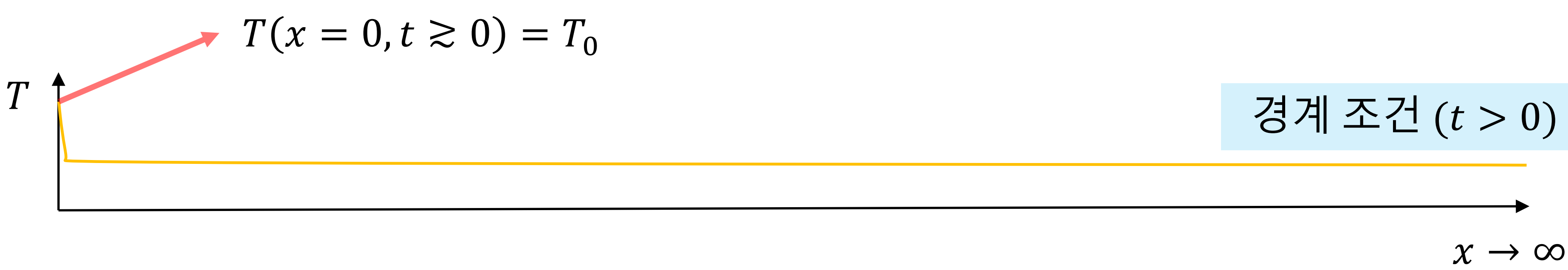
$$\frac{T - T_i}{(T_0 - T_i)} = 1 - \mathrm{erf}(\eta)$$

$$\frac{T - T_i}{(T_0 - T_i)} = 1 - \mathrm{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

초기 조건 $(t = 0)$




경계 조건 $(t > 0)$




T(x,t) in general?

$$\frac{T - T_i}{(T_0 - T_i)} = 1 - \mathrm{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

# 반무한 고체의 전도 예제 8.4

$T_i = 25°C,$
$T_0 = 100°C$
$\alpha = 1.2\times10^{-5} m^2/s$
반무한 steel rod에서
1차원 열전도식 응용.

$Q_1$ 5분후 봉의 한쪽면에서부터 0.1m 떨어진 지점의 온도는 얼마인가?

$Q_2$ 0.01m 떨어진 지점의 온도가 $75°C$ 로 상승하는데 걸리는 시간은 얼마인가?