Taylor Series

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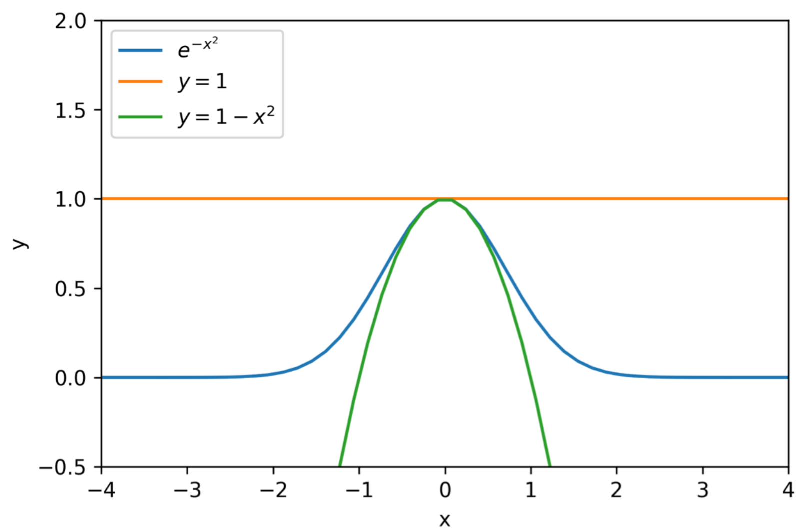
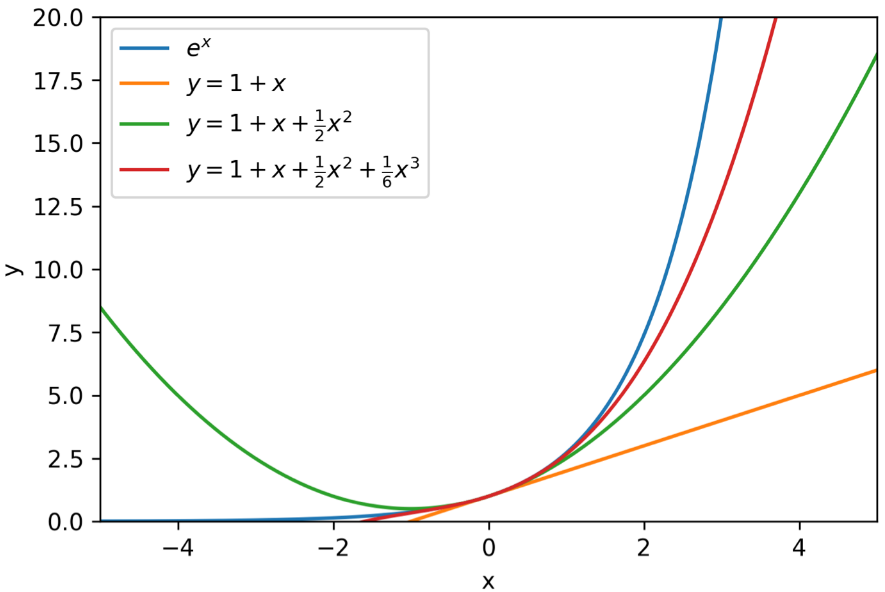
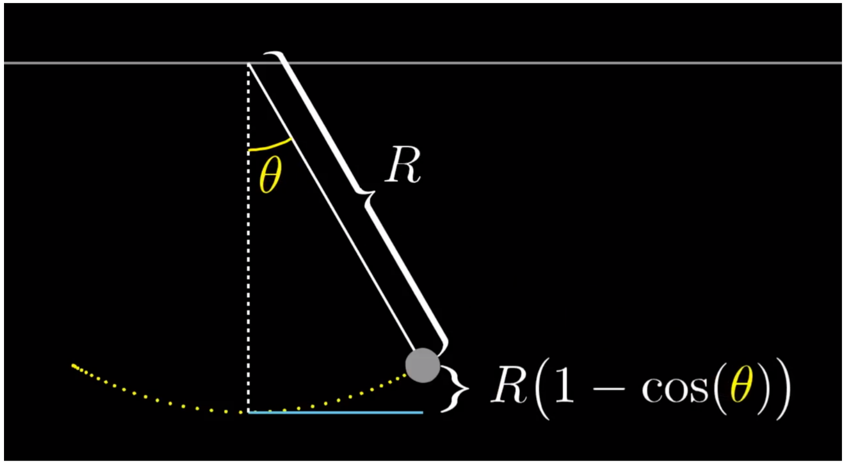
Taylor series (테일러 급수)





회전추의 높이



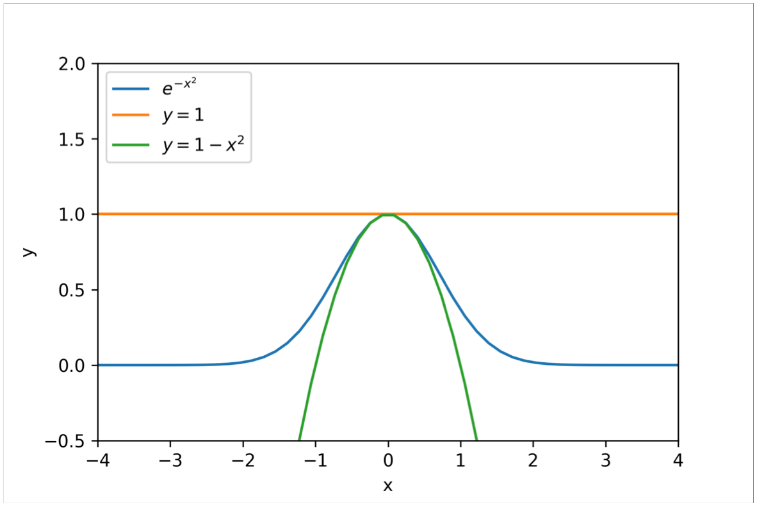
$$\cos(\theta) \approx 1 - \frac{\theta^2}{2}$$

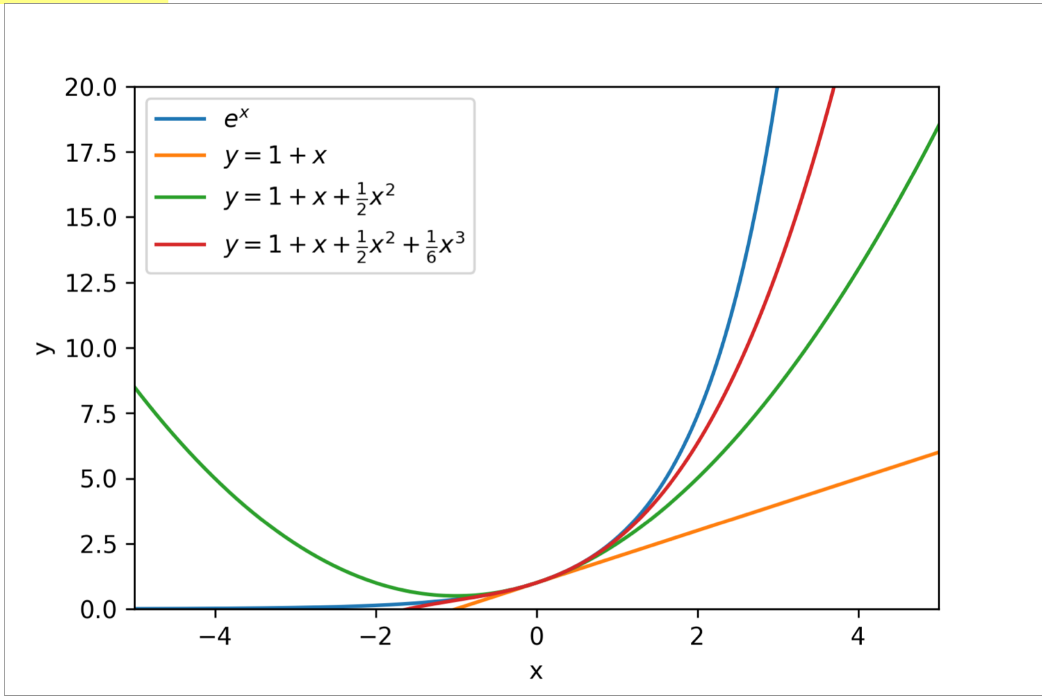
https://youtu.be/3d6DsjIBzJ4?t=38

Polynomial (다항식)

- Taylor series **approximate** an arbitrary function in the form of a polynomial

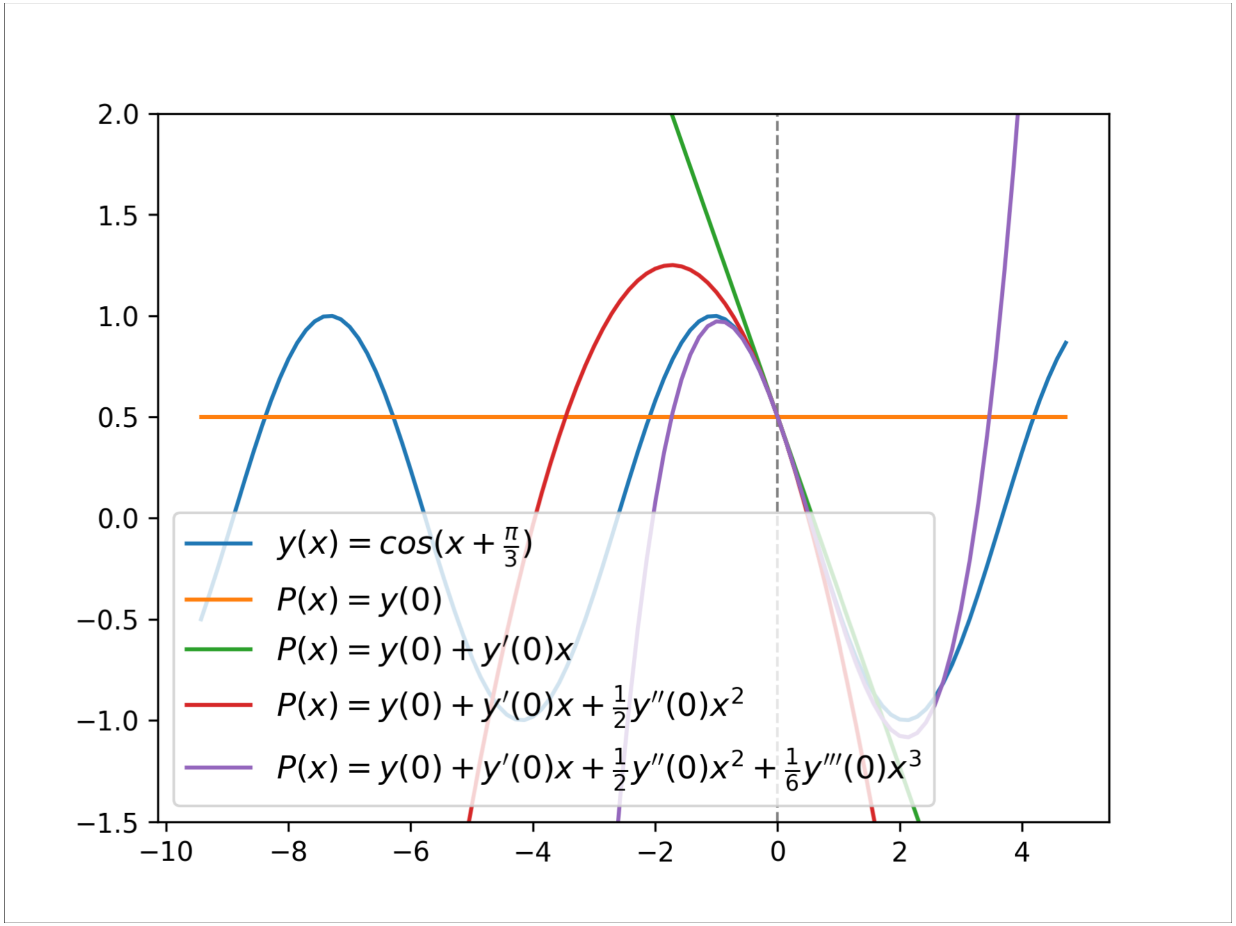
$$y(x) = a + bx + cx^2 + dx^3 \text{ ... and on on}$$





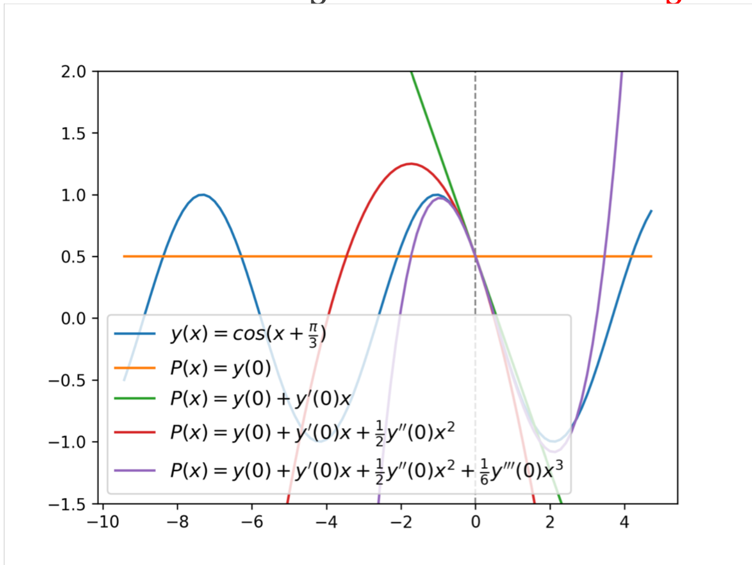
How to?

- Example of $y = \cos\left(x + \frac{\pi}{3}\right)$
- $P(x) = a + bx + cx^2$ ...



Verification

$$y(x) = \cos\left(x + \frac{\pi}{3}\right) \approx P(x) = \cos\left(0 + \frac{\pi}{3}\right) - \sin\left(0 + \frac{\pi}{3}\right)x - \frac{1}{2}\cos\left(0 + \frac{\pi}{3}\right)x^2 + \frac{1}{6}\sin\left(0 + \frac{\pi}{3}\right)x^3$$



$$y(0.1) = \cos\left(0.1 + \frac{\pi}{3}\right) = 0.41104380 \ \ldots$$

$$\cos\left(\frac{\pi}{3}\right) = 0.500000$$

$$\cos\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right) \times 0.1 = 0.413397 \ \ldots$$

$$\cos\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right) \times 0.1 - \frac{1}{2}\cos\left(\frac{\pi}{3}\right) \times 0.01 = 0.410897 \ \ldots$$

$$\cos\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right) \times 0.1 - \frac{1}{2}\cos\left(\frac{\pi}{3}\right) \times 0.01 + \frac{1}{6}\sin\left(\frac{\pi}{3}\right) \times 0.001 = 0.411041 \ \ldots$$

Generalization

$y = f(x)$

Taylor series of $f(x)$ at $x = a$?

$$= f(x = a) + \frac{df}{dx}(x = a)\times(x - a) + \frac{1}{2}\frac{d^2 f}{dx^2}(x = a)\times(x - a)^2$$

$$\frac{df}{dx}(x = a)$$

도함수 $\frac{df}{dx}$ 가 x의 함수라면, x에 a를 대입하여라.

통상, 2차항까지만 해도 제법 훌륭한 approximation 가능.

Numerical solution of ODE using Taylor series

$$\frac{dy(t)}{dt} = f(t,y)$$

Let's say we have a initial state such that

$$y(t=0) = y_0$$

$$= f(x=a) + \frac{df}{dx}(x=a)\times(x-a) + \frac{1}{2}\frac{d^2f}{dx^2}(x=a)\times(x-a)^2$$

$$y(x+a) = y(a) + y'(a)x + \frac{1}{2}y'(a)x^2 + \frac{1}{6}y''(a)x^3 \ \ldots$$

$$y(\Delta t + t) = ?$$

$$y(\Delta t + t) = y(t) + y'(t)\Delta t + \frac{1}{2}y''(t)(\Delta t)^2 + \frac{1}{6}y'''(t)(\Delta t)^3$$

$$y(\Delta t + t) = y(t) + y'(t)\Delta t + \frac{1}{2}y''(t)(\Delta t)^2$$

$$y'(t)\Delta t = y(\Delta t + t) - y(t) - \frac{1}{2}y''(t)(\Delta t)^2$$

$$y'(t) = \frac{y(\Delta t + t) - y(t)}{\Delta t} - \frac{1}{2}y''(t)\Delta t$$

$$\frac{dy(t)}{dt} = f(t,y) = y'(t)$$

This term decreases as Δt decreases.

$$\frac{dy(t)}{dt} = f(t,y) = y'(t) \approx \frac{y(\Delta t + t) - y(t)}{\Delta t} - \frac{1}{2}y''(t)\Delta t$$

$$f(t,y) = \frac{y(\Delta t + t) - y(t)}{\Delta t} - \frac{1}{2}y''(t)\Delta t$$

$$y(\Delta t + t) = \Delta t f(t,y) + y(t) + \frac{1}{2}y''(t)(\Delta t)^2$$

$$y(\Delta t + t) = y(t) + \Delta t f(t,y)$$

$$y(t_{n+1}) = y(t_n) + \Delta t f(t_n, y_n)$$

Taylor series and Euler method

$$\frac{dy(t)}{dt} = f(t, y)$$

Let's say we have a initial state such that

$$y(t = 0) = y_0$$

$$y(t_{n+1}) = y(t_n) + \Delta t f(t_n, y_n)$$

$$y(t_{n+1}) = y(t_n) + \Delta t \frac{dy(t_n)}{dt}$$

You know exactly what $\frac{dy(t_n)}{dt}$ is.

You don't know what $y(t)$ is.

Following the Euler method, once you know $y(t = 0) = y_0$, you can obtain $y(t = \Delta t) = y_1$ and so $y(t = 2\Delta t) = y_2$, and so forth.

Provided that 1) your approximation using the Taylor expansion is reasonable; and that 2) the time increment Δt is sufficiently small.