Macroscopic Plasticity and Yielding

강의명: 금속유동해석특론 (AMB2039)

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Recap

- Measurement of force and displacement from tension tests
- Physical quantity to remove the effect of geometry: engineering stress/engineering strain
- Two types of stress (strain):
  - Normal (tension + , or compression -)
  - Shear (forward +, backward -)
- There are three independent planes in 3D; On each plane 1 normal + 2 shears.
- Thus nine independent components comprise the stress (strain) state.
- Coordinate transformation (axes transformation)
  - Coordinate transformation does not change the physical quantity (stress, strain)
  - Coordinate transformation changes the values of components and the directions of planes associated with the stress (or strain).
- Practice coordinate transformation using the Excel, Fortran code, Python code.

Outline

- 소성변형을 해석하기 위한 현상학적 접근법
- 항복 조건 (탄성 변형을 넘어, 소성 변형 또한 발생하게 만드는 조건은 무엇인가?)

항복 조건 (yield condition; yield criterion)

- 항복 조건을 수학적으로 표현한다. 주로 변수는 '응력'으로 나타낸다.
- 앞서 다루었듯, 응력은 6개의 성분으로 구성된 응력 텐서로 표현가능 (힘평형)
- 따라서, 항복 조건을 따지는 함수 f 는 다음과 같이 표기 가능하다.

$$f = f(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12})$$

- 만약 응력 텐서를 참조하는 공간이 principal space라면 위를

$$f = f(\sigma_1, \sigma_2, \sigma_3)$$

- 로 표현할 수도 있다.
- 항복 조건은 대게 위의 항복 함수가 특정 값을 가질 때 만족하는 방식으로 표현하다. 따라서,

$f(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12}) = C$ (항복 조건 만족, 즉 소성변형률이 발생)
$f(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12}) < C$ (항복 조건 불만족, 즉, 탄성영역)

- 물론 principal space에서 위는 더욱 축약되어 표현될 수 있다.

$f(\sigma_1, \sigma_2, \sigma_3) = C$ (항복 조건 만족, 즉 소성변형률이 발생)
$f(\sigma_1, \sigma_2, \sigma_3) < C$ (항복 조건 불만족 즉, 탄성영역)

항복 함수와 등방성 금속들의 특징

- 항복 조건을 따지는 함수 f를 우리는 '항복 함수' (yield function)이라 부른다.

- 소성 영역에서 등방성을 띄는 금속들은 때때로 다음과 같은 특징을 띄는 것으로 가정한다.
  1. Bauschinger effect is negligible (인장시의 항복 강도와 압축시의 항복 강도가 동일)
  2. 소성 변형으로 인한 부피 변화가 없다 (탄성으로 인한 부피 변화는 가능)
  3. 평균수직응력 (mean value of normal stress components) 값은 항복에 영향을 주지 않는다.

$$\sigma_m = \left(\frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3}\right)$$

Stress tensor를 다른 좌표계로 변환시켜도 이 값은 변하지 않는다. 응력텐서에서 이렇게 좌표계에 무관한 세종류의 값을 얻을 수 있는데, 이를 invariant 라고 한다 (추후에 다루자.)

- 위의 세 가정은 소성등방성(plastic isotropy)를 가정한 재료에만 해당된다. 대부분의 재료는 소성이방성을 가진다. 비교적 간단한 등방성 재료 경우부터 배우기 시작해 더 복잡한 이방성재료로 확대하여 배우도록 하자.

Deviatoric stress

- 앞서 기술한 금속의 소성 특징중 다음 문장을 다시 살펴보자.
  - 평균수직응력 (average of normal stress components) 값은 항복에 영향을 주지 않는다.
- 위 금속의 소성 특성을 설명하는 수학적 모델에 매우 큰 영향을 끼친다.
  - 항복은 응력에 대한 함수다.
  - 평균 수직응력이 항복에 영향을 주지 않는다.
  - 따라서 항복 여부를 응력상태에서 판가름할 때 '평균수직응력' 성분은 고려하지 않아도 된다.
  - 그렇다면 항복상태 판가름할 때 기준이 되는 응력에서 평균수직 성분만 제외할 수 있을까?
- 응력에서 수직응력 성분만 '제외' 하여 새로운 종류의 응력 텐서를 표현해보자.

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} - \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} = \text{새로운 종류의 응력}$$

(stress deviator, deviatoric stress tensor)

$\boldsymbol{\sigma}_m = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}$: 평균수직응력텐서

(mean stress)

- 세 직각 방향으로 수직응력이 작용 (대기압, 수압)
- Stress transformation sheet로 실습해보자

Mean stress

2nd rank tensor in matrix form

100	0	0
0	100	0
0	0	100

	angle
phi1	10
Phi	55
phi2	-10

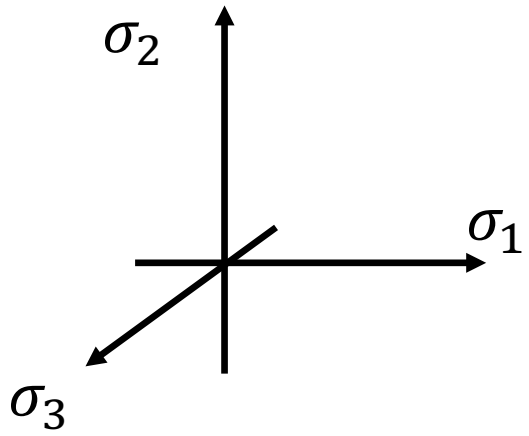
100.000	0.000	0.000
0.000	100.000	0.000
0.000	0.000	100.000

Put whatever rotation you'll get the same stress tensor.

Mean stress 형태의 응력을 hydrostatic pressure라고도 한다…

Yield surface

- 앞서 간략히 살펴본 항복 함수를 주응력 공간에서 표현하면
- $f(\sigma_1, \sigma_2, \sigma_3) = Constant$
- 로 표현할 수 있다.
- 이를 주응력 공간, 즉 세 주응력 성분 값들을 서로 수직한 세 축으로 표현한 수직좌표계 공간에 옮겨서 표현하면,



- $f(\sigma_1, \sigma_2, \sigma_3) = Constant$ 를 만족하는 '점'들의 모임은 '면'을 이루게 된다.
- 이렇게 항복 함수의 항복 조건을 만족시키는 응력 공간(stress space)에 표현된 '면'을 '항복면' (yield surface)이라 한다.

Yield surface

Given the stress space, the yield criterion can be defined as

$\bar{\sigma}(\sigma_{ij}) - \sigma_y = 0$ (plastic)

$\bar{\sigma}(\sigma_{ij}) - \sigma_y < 0$ (elastic)

σ_y: a scalar value that represents the onset of yielding.
In case of 'uniaxial' tension stress state, σ_y is the yield strength

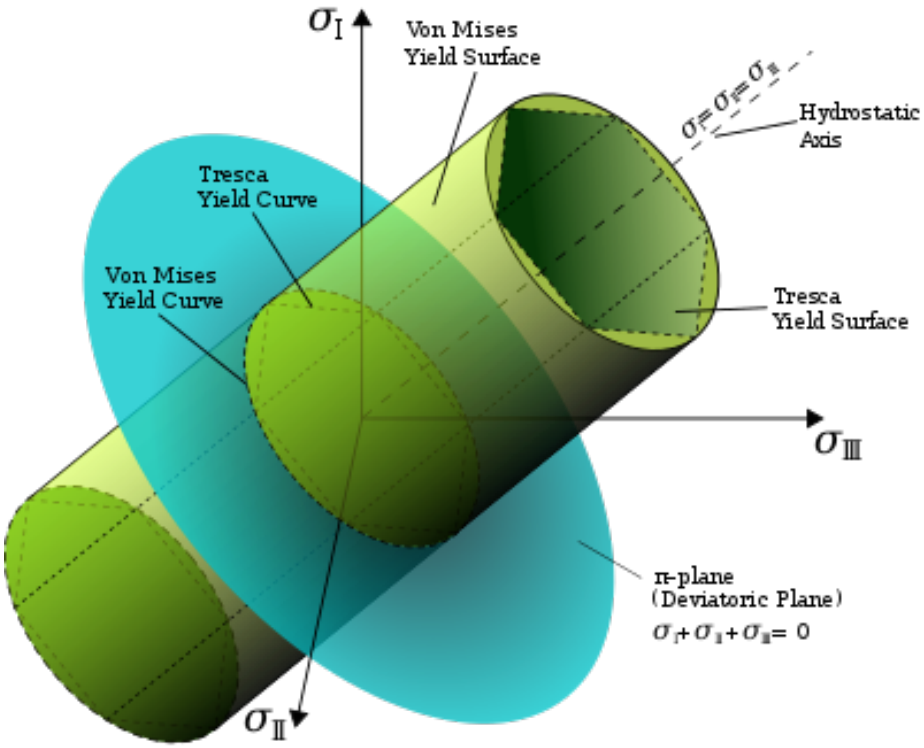
$\bar{\sigma}(\sigma_{ij})$ returns a scalar value as a function of stress tensor σ_{ij}

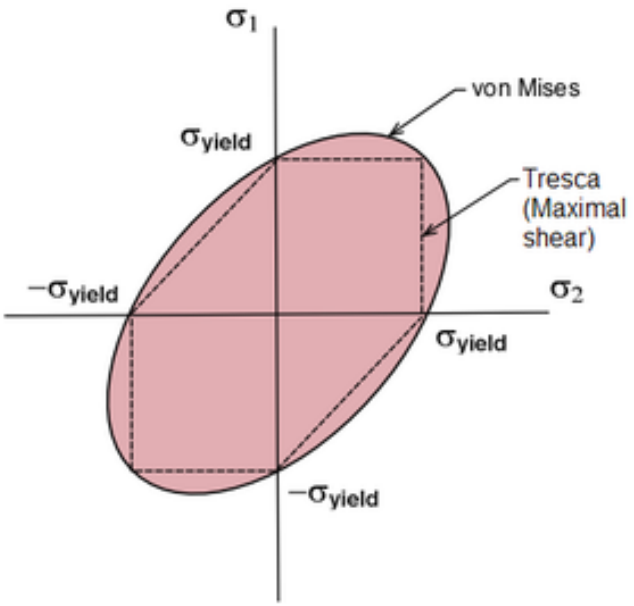
Collection of σ_{ij} points in stress space that satisfy the yield criteria $\bar{\sigma}(\sigma_{ij}) - \sigma_y = 0$ can be viewed in the form of:

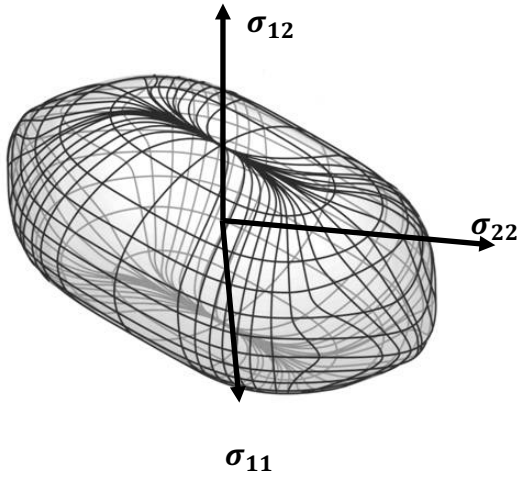
1. A sing point (uniaxial tension stress state, for example)
2. Locus (in the stress space having only two axes, e.g., σ_{11} and σ_{22}
3. 3D surface if the stress space consists of σ_{11}, σ_{22} and σ_{12} (easy to visualize?)
4. 6D ***surface*** $\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}$, and σ_{12} (difficult to visualize)

In many cases, the yield surface (i.e., collection of σ_{ij} points that satisfy $\bar{\sigma}(\sigma_{ij}) - \sigma_y = 0$) is mathematically constructed using a 'function' called ***yield function***.

Example of yield surface represented in various (subset of) stress spaces







https://en.wikipedia.org/wiki/Yield_surface

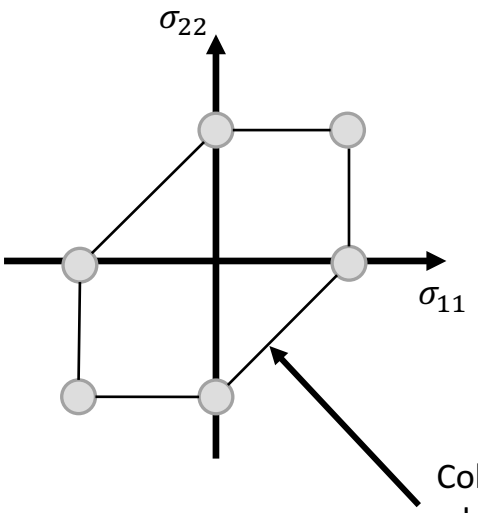
항복 현상과 coordinate transformation

- Yielding은 재료의 고유한 특징.
- 소성에서 등방적 거동을 하는 재료의 경우 가진 yield condition을 만족하는 응력은 coordinate transformation에 의존하지 않아야 한다 (차후에 실습해보자)
- 소성 영역에서 등방적 거동을 보이는 재료의 경우 다음 두가지 yield function이 대표적이다.
  - Tresca
  - Von Mises

Tresca I

$$\max\left(\left|\sigma^{I}-\sigma^{II}\right|,\left|\sigma^{II}-\sigma^{III}\right|,\left|\sigma^{III}-\sigma^{I}\right|\right)=\sigma_{\text{tresca}}\ (\text{상수})$$

σ_{tresca}: Tresca yield (scalar) value
$\sigma^{I},\sigma^{II},\sigma^{III}$: principal stresses



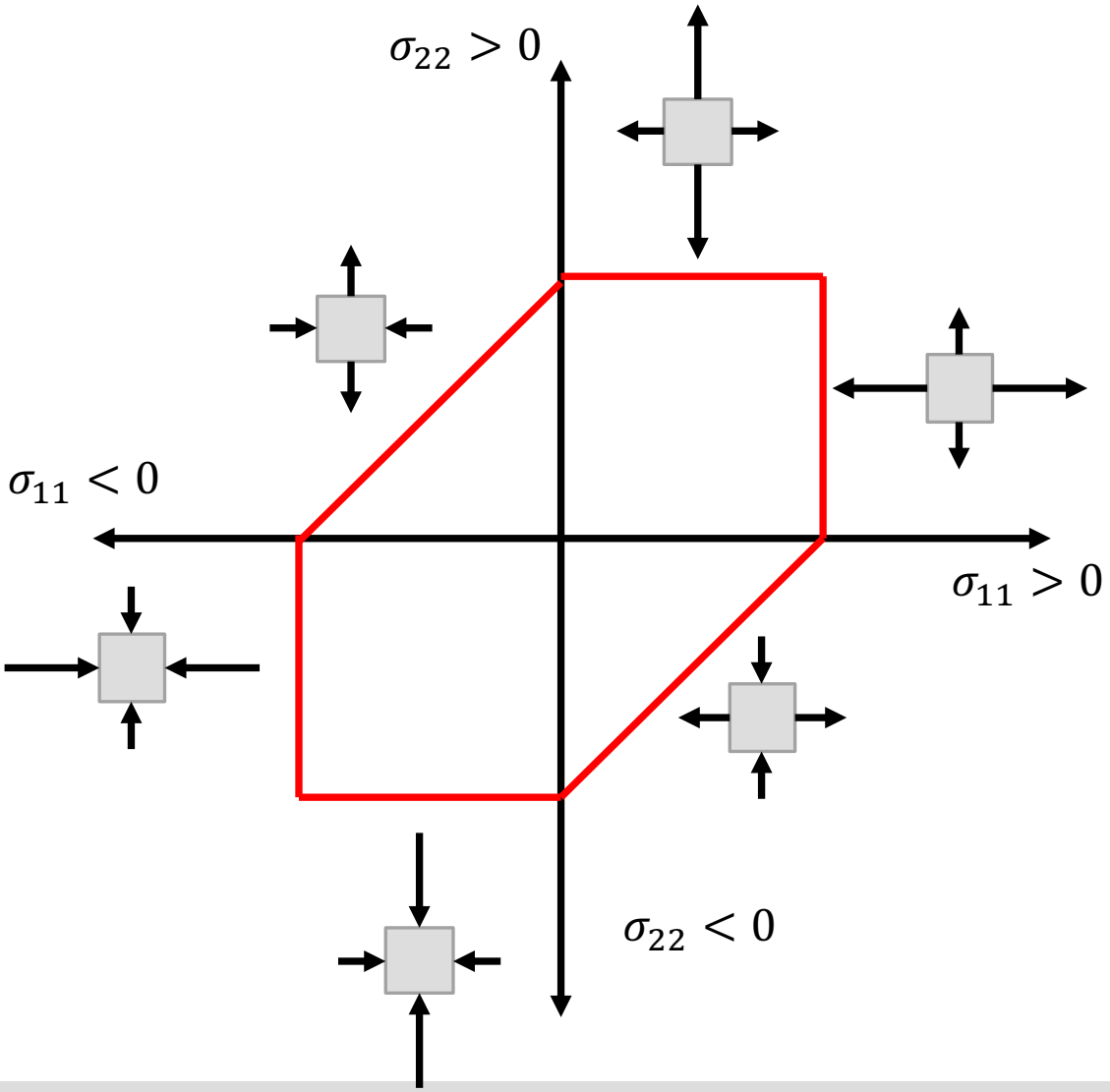
Collection of points where σ_{22} values are the same

$$\begin{bmatrix}\sigma_{11} & 0 & 0\\ 0 & \sigma_{22} & 0\\ 0 & 0 & 0\end{bmatrix}$$

Remember that in the Mohr circle representation, $|\sigma^{I}-\sigma^{II}|$ value is associated with the size of the circle. The radius of the Mohr circle is the maximum shear stress.

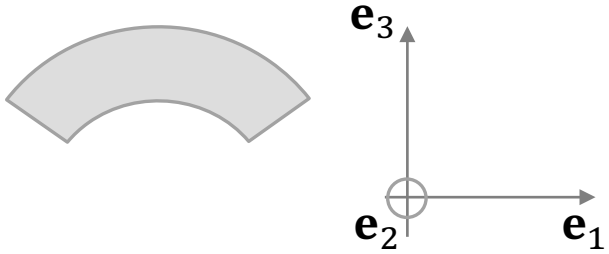
Tresca yield criterion is sometimes referred to as the maximum shear yield criterion.

Tresca II



예제 2-2

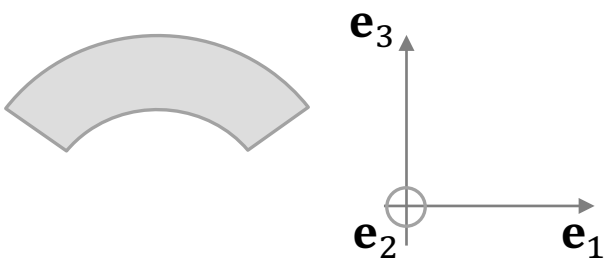
- 얇은 벽을 갖는 스테인레스 재질의 튜브에 고압의 가스를 보관하려고 한다. 튜브의 양 끝이 막혀 있으며 사용중 최대 20 MPa 의 내부 압력이 작용할 것으로 예상한다. 튜브의 직경은 약 20 cm 이며 어느 부분에서도 항복은 일어나지 않는다. 해당 튜브를 앞으로 10만개 생산할 예정인데, 최근 스테인레스의 가격이 Ni의 가격 상승으로 인해 매우 비싸졌다. 여러분들이 설계 엔지니어라면, 튜브 생산에 필요한 스테인레스 재질의 두께를 줄여 원가 절감을 원할 것이다.
- (a) 만약 이 재료의 인장 항복 강도 – 즉 일축 인장 응력 상태에서의 항복 강도 Y – 가 300 MPa 이고, Tresca의 항복 조건을 따른다고 가정할 때, 최소한의 튜브 두께는 얼마가 되어야 하는가?
- 풀이
- 앞서 예제 1-11 에서 같은 형태의 튜브에서 응력은 튜브내 압력 (p), 그리고 튜브의 반경(r), 그리고 두께(t)에 대한 함수로 표현된다.



$$\sigma = \begin{bmatrix} \frac{Pr}{t} & 0 & 0 \\ 0 & \frac{Pr}{2t} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ Is this principal space? Yes.

$$\sigma^{I} = \frac{Pr}{t}, \sigma^{II} = \frac{Pr}{2t}, \sigma^{III} = 0$$

예제 2-2 (계속)



$$\sigma = \begin{bmatrix} \frac{Pr}{t} & 0 & 0 \\ 0 & \frac{Pr}{2t} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ Is this principal space? Yes.

$$\sigma^{I} = \frac{Pr}{t}, \sigma^{II} = \frac{Pr}{2t}, \sigma^{III} = 0$$

- Tresca yield condition:

$$\max\left(|\sigma^{I} - \sigma^{II}|, |\sigma^{II} - \sigma^{III}|, |\sigma^{III} - \sigma^{I}|\right) = 300\ [\text{Mpa}]$$

$$|\sigma^{I} - \sigma^{II}| = \frac{Pr}{2t}$$
$$|\sigma^{II} - \sigma^{III}| = \frac{Pr}{2t}$$
$$|\sigma^{III} - \sigma^{I}| = \frac{Pr}{t}$$

P,r,t 모두 양수. 따라서 위 셋 중 max value는 $\frac{Pr}{t}$

항복이 발생하지 않을 조건:

$$\frac{Pr}{t} < 300\ [\text{Mpa}] \rightarrow \frac{Pr}{300[\text{MPa}]} < t$$

$$\frac{20\ [\text{MPa}] \times 20\ [\text{cm}]}{300[\text{MPa}]} < t$$

$$\therefore 1.3\ [\text{cm}] < t$$

따라서, 주어진 조건에서 여러분은 1.3[cm]보다 두꺼운 스테인레스 강을 사용해야 한다!

예제 2-2 (Based on deviatoric stress)

- $\sigma = \begin{bmatrix} \frac{Pr}{t} & 0 & 0 \\ 0 & \frac{Pr}{2t} & 0 \\ 0 & 0 & 0 \end{bmatrix}$ 에서

- $s = \begin{bmatrix} \frac{Pr}{t} & 0 & 0 \\ 0 & \frac{Pr}{2t} & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \frac{Pr}{2t} & 0 & 0 \\ 0 & \frac{Pr}{2t} & 0 \\ 0 & 0 & \frac{Pr}{2t} \end{bmatrix} = \begin{bmatrix} \frac{Pr}{2t} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{Pr}{2t} \end{bmatrix}$

$$\max\left(\left|s^{I} - s^{II}\right|, \left|s^{II} - s^{III}\right|, \left|s^{III} - s^{I}\right|\right) = 300 \text{ [Mpa]}$$

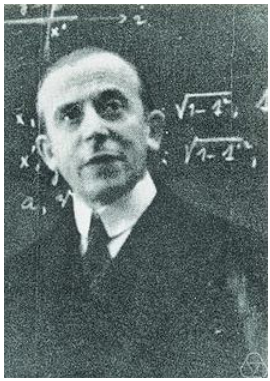
- $\max\left(\left|\frac{Pr}{2t}\right|, \left|\frac{Pr}{2t}\right|, \left|-\frac{Pr}{t}\right|\right) = 300$

- $\frac{Pr}{t} = 300$

- 따라서, 앞서 stress tensor를 사용할 때와 같은 결론에 이른다.

von Mises I

$$\max\left(\left|\sigma^{I}-\sigma^{II}\right|,\left|\sigma^{II}-\sigma^{III}\right|,\left|\sigma^{III}-\sigma^{I}\right|\right)=\sigma_{tresca}$$



von Mises postulates yield occurs when the root-mean-square shear stress reaches a critical value

$$\frac{\left(\sigma^{I}-\sigma^{II}\right)^{2}+\left(\sigma^{II}-\sigma^{III}\right)^{2}+\left(\sigma^{I}-\sigma^{III}\right)^{2}}{3}=C$$

If the uniaxial yield stress is Y, the principal stress value will be something like $\sigma^{I}=Y$, $\sigma^{II}=0$ and $\sigma^{III}=0$. In that case the VM criterion becomes:

$$\frac{2Y^{2}}{3}=C$$

$$\frac{\left(\sigma^{I}-\sigma^{II}\right)^{2}+\left(\sigma^{II}-\sigma^{III}\right)^{2}+\left(\sigma^{I}-\sigma^{III}\right)^{2}}{3}=\frac{2}{3}Y^{2}$$

Multiply 3 on both sides

$$\left(\sigma^{I}-\sigma^{II}\right)^{2}+\left(\sigma^{II}-\sigma^{III}\right)^{2}+\left(\sigma^{I}-\sigma^{III}\right)^{2}=2Y^{2}$$

von Mises II

$$\left(\sigma^{I}-\sigma^{II}\right)^{2}+\left(\sigma^{II}-\sigma^{III}\right)^{2}+\left(\sigma^{I}-\sigma^{III}\right)^{2}=2Y^{2}$$

With Y being the yield stress under uniaxial tension

In a general stress space where shear components may not be zero, the above becomes:

$$(\sigma_{11}-\sigma_{22})^{2}+(\sigma_{22}-\sigma_{33})^{2}+(\sigma_{33}-\sigma_{11})^{2}+6(\sigma_{12}^{2}+\sigma_{23}^{2}+\sigma_{13}^{2})=2Y^{2}$$

The above has many alternate forms. Among others, one based on deviatoric stress tensor **s** is very simple and useful:

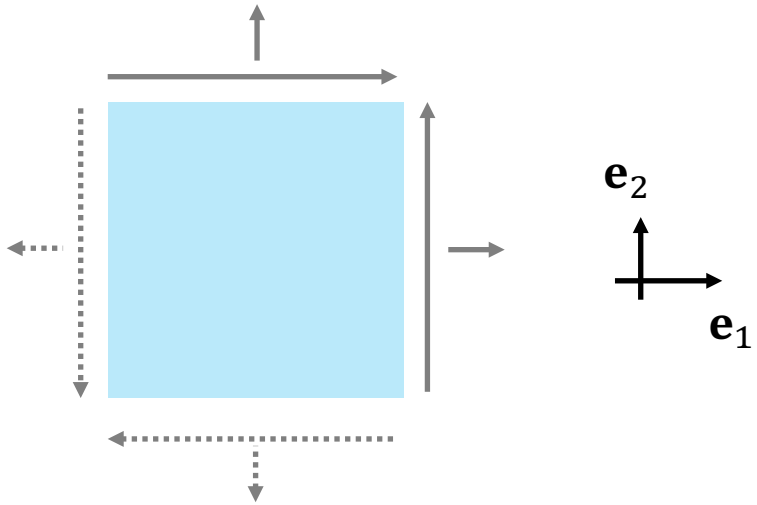
$$3\,\mathbf{s}\colon\mathbf{s}=3\,s_{ij}s_{ij}=2Y^{2}$$

von Mises III

3D stress tensor can be **too much** complicated. Often, problems can be approximated to be that under the plane-stress condition.

In 2D space, the stress tensor becomes …

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$$



$$(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2+6(\sigma_{12}^2+\sigma_{23}^2+\sigma_{13}^2)=2Y^2$$

2D space subspace

$$(\sigma_{11}-\sigma_{22})^2+\sigma_{22}^2+\sigma_{11}^2+6\sigma_{12}^2=2Y^2$$

if $\sigma_{12}=0$

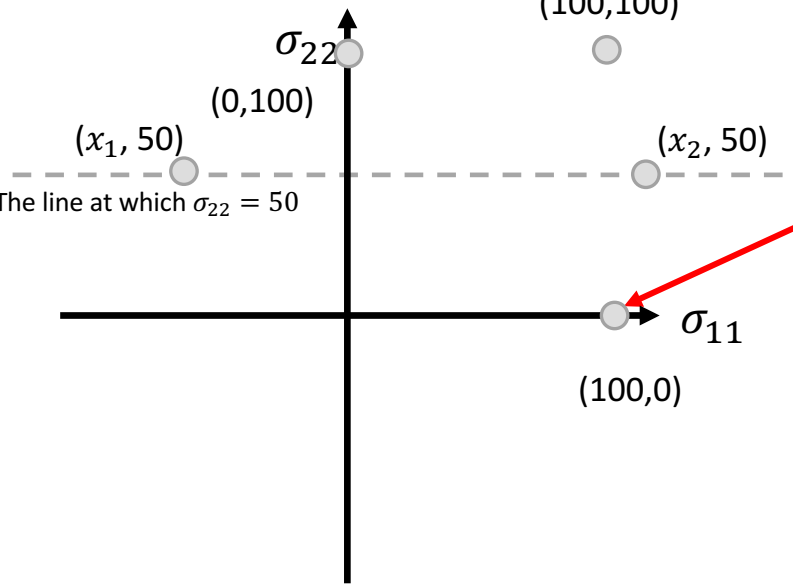
$$\sigma_{22}^2+\sigma_{11}^2-\sigma_{11}\sigma_{22}=Y^2$$

von Mises IV

$$\sigma_{22}^2 + \sigma_{11}^2 - \sigma_{11}\sigma_{22} = Y^2$$

$$\sqrt{\sigma_{22}^2 + \sigma_{11}^2 - \sigma_{11}\sigma_{22}} = Y$$

Suppose you have the uniaxial tension yield stress value of 100 MPa



Solve $\sqrt{50^2 + \sigma_{11}^2 - 50\sigma_{11}} = 100$

$$50^2 + \sigma_{11}^2 - 50\sigma_{11} = 100^2$$
$$\sigma_{11}^2 - 50\sigma_{11} + 25^2 = 100^2 - 50^2 + 25^2$$
$$(\sigma_{11} - 25)^2 = 150{\times}50 + 25^2$$

Therefore,

$$\sigma_{11} = \pm\sqrt{8125} + 25 = \pm 90.1 \ \ldots + 25$$

$$x_1 = -90.1 + 25$$
$$x_2 = +90.1 + 25$$

Tresca and von Mises 비교

- Follow below link
- https://youngung.github.io/yieldsurface/

Invariants

- 앞서 principal stress를 구했던 방법을 다시 생각해보자.

$$A = \begin{bmatrix} \sigma_{11} - \lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \lambda \end{bmatrix}$$

Where

$$\lambda^3 - I_1\lambda^2 - I_2\lambda - I_3 = 0$$

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

$$I_2 = (\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2 - \sigma_{11}\sigma_{22} - \sigma_{22}\sigma_{33} - \sigma_{33}\sigma_{11})$$

$$I_3 = \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{13}\sigma_{23} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{13}^2 - \sigma_{33}\sigma_{12}^2$$

- 여기서 I_1, I_2, I_3 값들은 좌표계를 바꾼다고 해서 바뀌지 않는다. 이 세 값을 invariants(불변량)라고 한다. 여기서 특히 $I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = 3\sigma_m$. 앞서 금속의 항복은 σ_m 값고 무관하다고 했으니, 항복은 또한 I_1 값과도 무관하다.

- 그림 2-5, 2-6 참고
- 예제 2-3과 2-4 참고

소성일 (plastic work)

- 일: 일이라는 물리량의 정의에서 단위를 살펴보면
  - 변위[m]x힘[N]; mN
- 소성일: 부피당 일 (specific work) 따라서 일/부피
  - 그 단위를 살펴보면
  - $\mathrm{m} \cdot \frac{\mathrm{N}}{\mathrm{m}^3} = [\mathrm{m}/\mathrm{m}] \cdot \mathrm{N}/\mathrm{m}^3$
- 소성일은 소성변형을 하는 물체에 가해진 응력과 그 반응인 변형률로 얻어지는 가해진 에너지의 총량을 의미한다.
- $\mathrm{dw} = \sigma_{ij} \mathrm{d}\varepsilon_{ij} = \Sigma_i^3 \Sigma_j^3 \sigma_{ij} \mathrm{d}\varepsilon_{ij}$
- 앞서 기술한 바와 같이 변형률과 응력이 '우연히' 동일한 principal space에 표현이 된다면…
- $\mathrm{dw} = \sigma_i \mathrm{d}\varepsilon_i = \sigma_1 \mathrm{d}\varepsilon_1 + \sigma_2 \mathrm{d}\varepsilon_2 + \sigma_3 \mathrm{d}\varepsilon_3$

유효응력 (effective/equivalent stress)

- 유효응력 (effective stress)은 응력의 함수 – 응력이 총 6개의 독립적인 성분을 가지므로, 그 변수는 총 6개 (물론 주응력공간에서는 줄어든다…)
- 유효응력은 많은 경우 앞서 다루었던 항복 함수 (yield function)과 같은 꼴을 가진다. 따라서 Tresca 항복 함수로 얻어지는 유효 응력은 Tresca 유효응력, 마찬가지로 von Mises 항복 함수로 얻어지는 유효 응력은 von Mises 유효응력이라고 일컫는다.
- 유효응력은 bar 를 사용해 σ로 표현. 따라서 $\bar{\sigma}^{\mathrm{Tresca}}, \bar{\sigma}^{\mathrm{VM}}$ 으로 위 두 경우를 구분.

- Principal stress space에서
- $\bar{\sigma}^{\mathrm{Tresca}} = |\sigma_1 - \sigma_3|$ when $\sigma_1 \geq \sigma_2 \geq \sigma_3$
- $\bar{\sigma}^{\mathrm{VM}} = \frac{1}{\sqrt{2}}\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\}^{1/2}$

유효변형률

- 유효 응력의 정의는 앞서 살펴본 것과 같이 항복함수를 빌려온다.
- (1)유효 변형률은 그러현 유효 응력으로부터 도출되는 소성일과 '짝'을 이룬다.
- 앞서 정의한 소성일을 다시 살펴보면
- (2) $dw = \sigma_{ij} d\varepsilon_{ij}$
- 위의 (1)은 다음의 성질을 말로 옮긴 것이다...
- (3) $dw = \bar{\sigma} d\bar{\varepsilon}$
- 따라서, (2)와 (3)으로
- $\sigma_{ij} d\varepsilon_{ij} = \bar{\sigma} d\bar{\varepsilon}$
- 그런데 앞에서 살펴보았듯이 $\bar{\sigma}$은 응력에 대한 함수다. 따라서 $\bar{\sigma}$을
- $\bar{\sigma} \equiv \bar{\sigma}(\sigma_{ij})$
- $\frac{\sigma_{ij} d\varepsilon_{ij}}{\bar{\sigma}} = d\bar{\varepsilon}$ 혹은 $\frac{dw}{\bar{\sigma}} = d\bar{\varepsilon}$

von Mises equivalent strain

- $\frac{\sigma_{ij} d\varepsilon_{ij}}{\bar{\sigma}^{VM}} = d\bar{\varepsilon}^{VM}$
- 만약 응력과 변형률이 모두 동일한 principal space에 참조된다면
- $\frac{\sum_i^3 \sigma_i d\varepsilon_i}{\bar{\sigma}^{VM}} = d\bar{\varepsilon}^{VM}$
- von Mises 유효 응력의 정의: $\bar{\sigma}^{VM} = \frac{1}{\sqrt{2}}\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\}^{1/2}$
- 따라서 … 복잡한 절차를 거쳐서
- $d\bar{\varepsilon}^{VM} = \frac{\sqrt{2}}{3}\{(d\varepsilon_1 - d\varepsilon_2)^2 + (d\varepsilon_2 - d\varepsilon_3)^2 + (d\varepsilon_3 - d\varepsilon_1)^2\}^{1/2}$

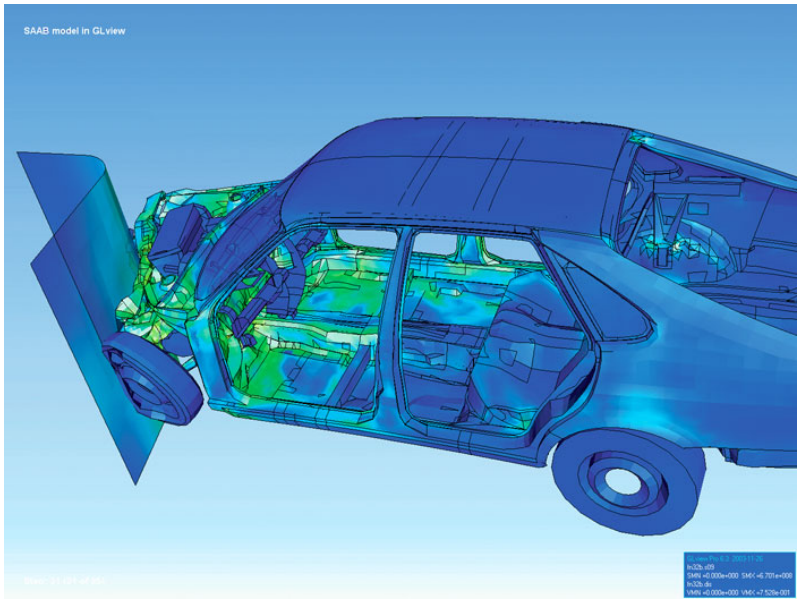
$$= \left[\frac{2}{3}(d\varepsilon_1^2 + d\varepsilon_2^2 + d\varepsilon_3^2)\right]^{1/2}$$

- 변형이 비례적으로 발생하는 소성변형후의 총유효 VM 변형률은
- $\bar{\varepsilon}^{VM} = \left[\frac{2}{3}(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2)\right]^{1/2}$
- 소성변형간 변형률의 변화가 각 구성성분간 비례적이지 않다면 직접 $d\bar{\varepsilon}^{VM}$ 값들을 경로 적분해야 합니다.

유효 응력과 유효 변형률의 쓰임

- 1) 앞서 변형률과 응력이 역학에서 여러 성분을 바탕으로 표현되는 것을 배웠다. 하지만 때때로 이렇게 텐서의 형태를 유지하며 표현하기에 복잡할 때가 있다. 대표적인 예가 (앞으로 다룰) 변형 경화 모델이다. 많은 철강 재료들의 변형경화 방식은 Hollomon equation 이라고 불리는 power-law 타입의 형태로 종종 표현된다:
- $\sigma = k\varepsilon^n$
- 여러분들은 위 공식을 일축 인장에서 얻어진 응력/변형률 곡선에 적용해 왔음을 배웠을 것이다. 하지만 가공 경화는 응력상태(혹은 변형률 상태)에 상관없이 발생한다. 이를 위해 유효 응력과 유효 변형률을 적절히 사용할 수 있다.
- $\bar{\sigma} = k\bar{\varepsilon}^n$

복잡한 형상의 구조물에서 응력이나 변형률을 텐서의 형태로 시각화하기란 쉽지 않다. 하지만 유효응력과 유효 변형률은 scalar value, 따라서 응력과 변형률의 '크기'를 시각화하기 편리하다. (왼편의 경우, 유효 응력값을 color-mapping)



https://en.wikiversity.org/wiki/Nonlinear_finite_elements

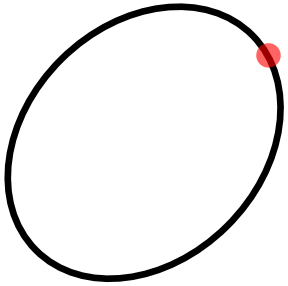
Yield and Flow

When the stress state of a material on the yield surface, the material plastically flows.

- Material (plastic) flow means that there is a certain amount (and type) of (plastic) strain
- This plastic flow occurs *instantaneously*: it occurs as soon as the stress state of the material reaches the yield point (locus, surface …); As soon as the material satisfies the yield criterion.
- How, then, does the material flow? In which direction? And what will be its magnitude along a particular direction?
- The answer is described by so-called the flow theory: theory that describes the flow behavior of materials.

Yield and Flow

If a unit volume of material is under a load sufficient to cause plastic deformation (σ_{ij}^{flow}), the material responds in terms of plastic strain increment ($d\varepsilon_{ij}^{pl}$) – that's how we deal with 'instantaneous' nature found in the material's responsive behavior to mechanical stimulus (in this case usually stress/force).



Yield surface

- **Flow stress** (σ_{ij}^{flow}): stress required to make the material continue plastically deforming
- Therefore, when the stress state of material is on the yield surface, the material flows (thus $d\varepsilon_{ij}^{pl} \neq 0$)
- At that instant of material flow, the incremental work done is then:

$$dw^{pl} = \sigma_{ij}^{flow} d\varepsilon_{ij}^{pl}$$

σ_{ij}^{flow} the flow stress may depend on the amount of plastic strain applied to the material

→ The flow stress usually increases as strain increases

Strain hardening

Metal flow theory I

$\varepsilon_{ij}^{total} = \varepsilon_{ij}^{pl} + \varepsilon_{ij}^{el}$ 전체 변형률은 plastic strain + elastic strain (additive decomposition)

탄성 구간에서의 flow : Hooke's law

$$d\sigma_{ij} = \mathbb{E}_{ijkl}\, d\varepsilon_{kl}^{el} \qquad d\varepsilon_{ij}^{el} = \mathbb{E}_{ijkl}^{-1}\, d\sigma_{kl}$$

$\mathbb{E}_{ijkl}$: 4th rank elastic moduli tensor

소성 구간에서의 flow: (plastic) flow theory

$$d\varepsilon_{ij}^{pl} = d\lambda \frac{\partial \phi^{pl}(\boldsymbol{\sigma})}{\partial \sigma_{ij}}$$

ϕ^{pl} is the *plastic potential* (defined in the stress space)

$d\lambda$: plastic multiplier (in the incremental form) usually is a constant

$d\varepsilon_{ij}^{pl}$ incremental plastic strain

$\frac{\partial \phi^{pl}(\boldsymbol{\sigma})}{\partial \sigma_{ij}}$: normal direction vector attached to the plastic potential ϕ^{pl} at the stress state of σ_{ij}

Metal flow theory II

$\frac{\partial \phi^{pl}(\boldsymbol{\sigma})}{\partial \sigma_{ij}}$

$\phi^{pl}(\boldsymbol{\sigma})$ the plastic potential is determined by a certain rule called “flow rule”

$\frac{\partial \phi^{pl}(\boldsymbol{\sigma})}{\partial \sigma_{ij}}$ can be also written in the principal stress space such that $\frac{\partial \phi^{pl}(\boldsymbol{\sigma})}{\partial \sigma_{i}}$ with the index *i* being 1,2, and 3

Thus, in the principal stress space, $\frac{\partial \phi^{pl}(\boldsymbol{\sigma})}{\partial \sigma_{ij}}$ becomes $\frac{\partial \phi^{pl}(\sigma_1,\sigma_2,\sigma_3)}{\partial \sigma_{i}}$.

$\frac{\partial \phi^{pl}(\sigma_1,\sigma_2,\sigma_3)}{\partial \sigma_{i}}$ actually is a three dimensional **direction vector** attached to the plastic potential; One very important observation is that such a direction is **normal** (perpendicular) to the potential.

Incompressible materials (such as metals) are well described by the **associated flow rule**, in which the plastic potential is described by a homogeneous **yield function**

Metal flow theory with homogeneous function

Then, what is a homogeneous function? What properties should we know?

A homogeneous function (*f*) of degree *n* in the space of (x,y) obeys:

$f(tx, ty) = t^n f(x, y)$ where *t* is any arbitrary constant

Function *f* is called as a homogeneous function of degree *n*

Euler's theorem on the Homogeneous function of degree *n*

$$x\frac{\partial f(x,y)}{\partial x} + y\frac{\partial f(x,y)}{\partial y} = n\, f(x,y)$$

→ Can be extended to a bigger dimensional space

If there is a homogeneous function constructed in the stress space of (σ_{11}, σ_{22}, σ_{12}):

$$\sigma_{11}\frac{\partial f(\sigma_{11},\sigma_{22},\sigma_{12})}{\partial \sigma_{11}} + \sigma_{22}\frac{\partial f(\sigma_{11},\sigma_{22},\sigma_{12})}{\partial \sigma_{22}} + \sigma_{12}\frac{\partial f(\sigma_{11},\sigma_{22},\sigma_{12})}{\partial \sigma_{12}} + \sigma_{12}\frac{\partial f(\sigma_{11},\sigma_{22},\sigma_{12})}{\partial \sigma_{12}} = n\, f(\sigma_{11},\sigma_{22},\sigma_{12}):$$

a homogeneous function f of degree n in the space of (σ_{11},σ_{22}, σ_{12})

If the homogeneous function in the space of ($\boldsymbol{\sigma}$) has the degree of 1:

$$\sigma_{ij}\frac{\partial f(\boldsymbol{\sigma})}{\partial \sigma_{ij}} = f(\boldsymbol{\sigma})$$

Metal flow theory II

$\frac{\partial \phi^{pl}(\boldsymbol{\sigma})}{\partial \sigma_{ij}}$ We assume ϕ^{pl} to be a homogeneous yield function of degree one in the stress space (σ_{ij})

From the general form of the flow rule:

$d\varepsilon_{ij}^{pl} = d\lambda \frac{\partial \phi^{pl}(\boldsymbol{\sigma})}{\partial \sigma_{ij}}$ No summation symbols missing $d\varepsilon_{11}^{pl} = d\lambda \frac{\partial \phi^{pl}(\boldsymbol{\sigma})}{\partial \sigma_{11}}$, $d\varepsilon_{22}^{pl} = d\lambda \frac{\partial \phi^{pl}(\boldsymbol{\sigma})}{\partial \sigma_{22}}$, $d\varepsilon_{33}^{pl} = d\lambda \frac{\partial \phi^{pl}(\boldsymbol{\sigma})}{\partial \sigma_{33}}$ …

If we multiply σ_{ij} on the left and the right hand sides,

$\sigma_{kl} d\varepsilon_{kl}^{pl} = d\lambda \sigma_{ij} \frac{\partial \phi^{pl}(\boldsymbol{\sigma})}{\partial \sigma_{ij}}$ Do not forget the missing summation symbols!

$\sigma_{kl} d\varepsilon_{kl}^{pl} = d\lambda \sigma_{ij} \frac{\partial \phi^{pl}(\boldsymbol{\sigma})}{\partial \sigma_{ij}} = d\lambda\, \phi^{pl}$

$\boldsymbol{\sigma} : d\boldsymbol{\varepsilon}^{pl} = dw^{pl}$

If we use a yield function for ϕ^{pl}, ϕ^{pl} can be treated as an equivalent stress.

In that case,

$d\lambda\, \phi^{pl} = d\lambda\, \sigma^{eq} = dw^{pl}$

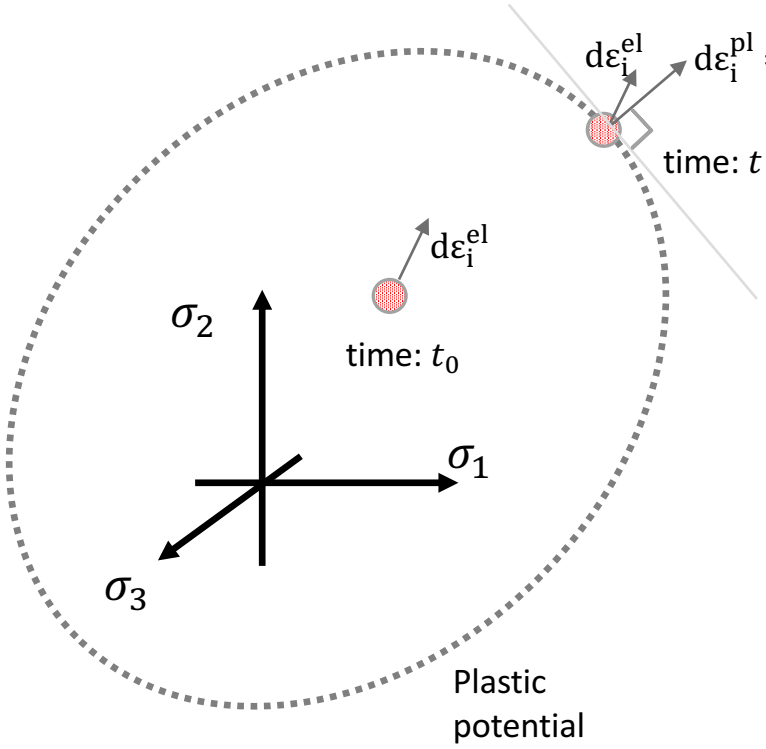
따라서 $d\lambda = d\varepsilon^{eq}$

Metal flow theory III (normality rule)

$$d\varepsilon_{ij}^{pl} = d\lambda \frac{\partial \phi^{pl}(\boldsymbol{\sigma})}{\partial \sigma_{ij}}$$

In the principal space of strain and stress tensors

$$d\varepsilon_{i} = d\lambda \frac{\partial \phi^{pl}(\boldsymbol{\sigma})}{\partial \sigma_{i}}$$ with i=1,2,3 (again no summation)



Incompressible materials (such as metals) are well described by the **associated flow rule**, in which the plastic potential is described by a **yield surface**

Associated flow rule is sometimes called 'normality' rule

The term **associated** is originated from the fact that the principal spaces of the flow stress and plastic strain increment tensors are 'co-axial'. The basis vectors of each coordinate system are associated with each other (in other words, the two coordinate axes are 'aligned').

Associated flow rule and plastic strain ratio

$$\sqrt{\frac{(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2+6(\sigma_{12}^2+\sigma_{23}^2+\sigma_{13}^2)}{2}}=\phi^{\mathrm{vm}}$$

If your material follows von Mises yield criterion and and the **associated flow rule**:

$$\Delta\varepsilon^{\mathrm{vm,eq}}\frac{\partial\phi^{\mathrm{vm}}}{\partial\sigma_{11}}=\mathrm{d}\varepsilon_{11}$$

$\varepsilon^{\mathrm{vm,eq}}$: von Mises equivalent strain

In the principal stress space:

$$\sqrt{\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}}=\phi^{\mathrm{vm}}$$

$$\Delta\varepsilon^{\mathrm{vm,eq}}\frac{\partial\phi^{\mathrm{vm}}}{\partial\sigma_1}=\mathrm{d}\varepsilon_1$$

Principal space of strain is co-axial to that of stress.

Associated flow rule and plastic strain ratio

$$\Delta\varepsilon^{vm,eq}\frac{\partial\phi^{vm}}{\partial\sigma_i}=d\varepsilon_i$$ Principal space of strain is co-axial to that of stress

$$\sqrt{\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}}=\phi^{vm}$$

The above can be expressed as $X^{\frac{1}{2}}=\phi^{vm}$ where $X=\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}$

Chain rule: when $z=f(y)$ and $y=g(x)$ and if you want to obtain $\frac{\partial z}{\partial x}$, you can use below relation:

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial y}\frac{\partial y}{\partial x}$$

$$\frac{\partial\phi^{vm}}{\partial\sigma_1}=\frac{\partial\phi^{vm}}{\partial X}\frac{\partial X}{\partial\sigma_1}=\frac{1}{2}X^{-\frac{1}{2}}\left[\frac{2\sigma_1-2\sigma_2+2\sigma_1-2\sigma_3}{2}\right]=X^{-\frac{1}{2}}\left[\frac{2\sigma_1-\sigma_2-\sigma_3}{2}\right]=\frac{1}{\phi^{vm}}\left[\frac{2\sigma_1-\sigma_2-\sigma_3}{2}\right]$$

$$\frac{\partial\phi^{vm}}{\partial\sigma_2}=\frac{\partial\phi^{vm}}{\partial X}\frac{\partial X}{\partial\sigma_2}=\frac{1}{2}X^{-\frac{1}{2}}\left[\frac{2\sigma_2-2\sigma_1+2\sigma_2-2\sigma_3}{2}\right]=X^{-\frac{1}{2}}\left[\frac{2\sigma_2-\sigma_1-\sigma_3}{2}\right]=\frac{1}{\phi^{vm}}\left[\frac{2\sigma_2-\sigma_1-\sigma_3}{2}\right]$$

$$\frac{\partial\phi^{vm}}{\partial\sigma_3}=\frac{1}{\phi^{vm}}\left[\frac{2\sigma_3-\sigma_1-\sigma_2}{2}\right]$$

Associated flow rule and plastic strain ratio

$\Delta\varepsilon^{vm,eq}\frac{\partial\phi^{vm}}{\partial\sigma_i} = d\varepsilon_i$ index *i* denotes each basis vector (or axis) of the principal space.

If the material's stress state is $\begin{bmatrix} 100 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and the material plastically flows, what is the incremental form of plastic strain tensor?

Assumption: material follows the von Mises yield criterion and the associated flow rule:

$$d\varepsilon_1 = d\varepsilon^{vm,eq}\frac{\partial\phi^{vm}}{\partial\sigma_1} = d\varepsilon^{vm,eq}\frac{1}{\phi^{vm}}\left[\frac{2\sigma_1-\sigma_2-\sigma_3}{2}\right] = \frac{d\varepsilon^{vm,eq}}{100}\left[\frac{2\times100}{2}\right] = d\varepsilon^{vm,eq}$$

$$\phi^{vm} = \sqrt{\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}}$$

$$d\varepsilon_2 = d\varepsilon^{vm,eq}\frac{1}{\phi^{vm}}\left[\frac{2\sigma_2-\sigma_1-\sigma_3}{2}\right] = \frac{d\varepsilon^{vm,eq}}{100}\left[\frac{-100}{2}\right] = -\frac{d\varepsilon^{vm,eq}}{2}$$

$$d\varepsilon_3 = d\varepsilon^{vm,eq}\frac{1}{\phi^{vm}}\left[\frac{2\sigma_3-\sigma_1-\sigma_2}{2}\right] = \frac{d\varepsilon^{vm,eq}}{100}\left[\frac{-100}{2}\right] = -\frac{d\varepsilon^{vm,eq}}{2}$$

Can we obtain $d\varepsilon^{vm,eq}$?

$$d\varepsilon^{vm,eq} = \frac{dw^{pl}}{\sigma^{vm,eq}} = \frac{\sigma_{ij}d\varepsilon_{ij}^{pl}}{\sigma^{vm,eq}} = \frac{\sigma_k d\varepsilon_k^{pl}}{\sigma^{vm,eq}}$$

$d\varepsilon_1^{pl}, d\varepsilon_2^{pl}, d\varepsilon_3^{pl}$

Associated flow rule and plastic strain ratio

$\Delta\varepsilon^{vm,eq}\frac{\partial\phi^{vm}}{\partial\sigma_i} = d\varepsilon_i$ index *i* denotes each basis vector (or axis) of the principal space.

If the material's stress state is $\begin{bmatrix} 100 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and the material plastically flows, what is the incremental form of plastic strain tensor?

Assumption: material follows the von Mises yield criterion and the associated flow rule:

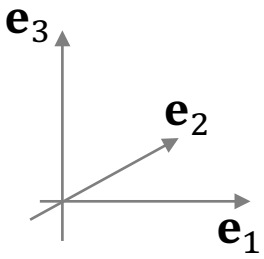
$$d\varepsilon_1 = d\varepsilon^{vm,eq}\frac{\partial\phi^{vm}}{\partial\sigma_1} = d\varepsilon^{vm,eq}\frac{1}{\phi^{vm}}\left[\frac{2\sigma_1-\sigma_2-\sigma_3}{2}\right] = \frac{d\varepsilon^{vm,eq}}{100}\left[\frac{2\times100-100}{2}\right] = \frac{d\varepsilon^{vm,eq}}{2}$$

$$\phi^{vm} = \sqrt{\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}}$$

$$d\varepsilon_2 = d\varepsilon^{vm,eq}\frac{1}{\phi^{vm}}\left[\frac{2\sigma_2-\sigma_1-\sigma_3}{2}\right] = \frac{d\varepsilon^{vm,eq}}{100}\left[\frac{2\times100-100}{2}\right] = \frac{d\varepsilon^{vm,eq}}{2}$$

$$d\varepsilon_3 = d\varepsilon^{vm,eq}\frac{1}{\phi^{vm}}\left[\frac{2\sigma_3-\sigma_1-\sigma_2}{2}\right] = \frac{d\varepsilon^{vm,eq}}{100}\left[\frac{-200}{2}\right] = -d\varepsilon^{vm,eq}$$

예제 2-7



$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

위 얇은 금속 판재가 2축 인장 응력 상태하에서 소성 변형률 비율이 $\varepsilon_{22}^{\mathrm{pl}} = -\frac{1}{4}\varepsilon_{11}^{\mathrm{pl}}$ 으로 측정되었다. Normality rule (associated flow rule)을 사용하여 해당 변형률이 발생할 수 있는 응력상태에서의 σ_{22}/σ_{11}의 비를 각각 von Mises 그리고 Tresca를 적용하여 구하여라

1) von Mises 풀이:

해당 응력 상태에서 $\phi^{\mathrm{VM}} = \sqrt{\frac{(\sigma_{11}-\sigma_2)^2+(\sigma_{22})^2+(-\sigma_1)^2}{2}}$

$\mathrm{d}\varepsilon_{11}^{\mathrm{pl}} = \mathrm{d}\lambda\frac{\mathrm{d}\phi^{\mathrm{VM}}}{\mathrm{d}\sigma_{11}}$ 그리고 $\mathrm{d}\varepsilon_{22}^{\mathrm{pl}} = \mathrm{d}\lambda\frac{\mathrm{d}\phi^{\mathrm{VM}}}{\mathrm{d}\sigma_{22}}$. $d\lambda$ 값은 공통의 상수이므로, 두 미소변형률의 비와의 관계를 위해 $\frac{\mathrm{d}\phi^{\mathrm{VM}}}{\mathrm{d}\sigma_{11}}$ 과 $\frac{\mathrm{d}\phi^{\mathrm{VM}}}{\mathrm{d}\sigma_{22}}$ 을 각각 구하면 되겠다: 이는 각각 $\frac{2\sigma_{11}-\sigma_{22}}{2}$, $\frac{2\sigma_{22}-\sigma_{11}}{2}$

따라서, $\frac{\mathrm{d}\varepsilon_{11}^{\mathrm{pl}}}{d\varepsilon_{22}^{pl}} = \left(\frac{2\sigma_{11}-\sigma_{22}}{2}\right)/\left(\frac{2\sigma_{22}-\sigma_{11}}{2}\right) = \frac{2\sigma_{11}-\sigma_{22}}{2\sigma_{22}-\sigma_{11}} = \frac{2-\sigma_{22}/\sigma_{11}}{2\sigma_{22}/\sigma_{11}-1} = -4$

$\rightarrow$ Solve following for x: $2 - x = -4(2x - 1) \rightarrow 7x = 2 \therefore x = 2/7$

References and acknowledgements

- References
  - An introduction to Continuum Mechanics – M. E. Gurtin
  - Metal Forming – W.F. Hosford, R. M. Caddell (번역판: 금속 소성 가공 - 허무영)
  - Fundamentals of metal forming (R. H. Wagoner, J-L Chenot)
  - http://www.continuummechanics.org (very good on-line reference)
- Acknowledgements
  - Some images presented in this lecture materials were collected from Wikipedia.