Elasticity required when studying plasticity

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Constitutive description on elasticity

Elastic constitutive law:

$\mathbb{E}\varepsilon = \sigma$ (elastic stiffness $\mathbb{E} = 200$ [GPa])

Elastic constitutive law:

$$\mathbb{E}_{ijkl}\varepsilon_{kl} = \sigma_{ij}$$

Apply it to FORTRAN, (or Python or Excel)

-Exercise 1. [3x1] = [3x3] [3x1]

-Exercise 2. [nx1] = [nxn] [nx1]

- (hint): use $[A]_i = [B]_{ij}[C]_j$

Kronecker delta may appear in formula

$$a = \varepsilon_{kl}\delta_{kl} = \sum_k \sum_l \varepsilon_{kl}\delta_{kl} = \sum_l \varepsilon_{1l}\delta_{1l} + \sum_l \varepsilon_{2l}\delta_{2l} + \sum_l \varepsilon_{3l}\delta_{3l}$$

$$= \varepsilon_{11}\delta_{11} + \varepsilon_{22}\delta_{22} + \varepsilon_{33}\delta_{33} = \sum_k \varepsilon_{kk} = \sum_i \varepsilon_{ii} = \sum_l \varepsilon_{ll}$$

$$= \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

Linear isotropic elasticity

Elastic constitutive law (Hooke's law):

$$\mathbb{E}_{ijkl}\varepsilon_{kl} = \sigma_{ij} \qquad (linear\ elasticity)$$

$$\mathbb{E}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \qquad (isotropic\ elasticity;\ two\ constants\ \lambda, \mu)$$

Replacing $\mathbb{E}_{ijkl}$ to the Hooke's law

$$\begin{aligned}\sigma_{ij} &= \mathbb{E}_{ijkl}\varepsilon_{kl} = \lambda\delta_{ij}\delta_{kl}\varepsilon_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\varepsilon_{kl} \\ &= \lambda\delta_{ij}\varepsilon_{kk} + \mu\left(\delta_{ik}\delta_{jl}\varepsilon_{kl} + \delta_{il}\delta_{jk}\varepsilon_{kl}\right) = \lambda\delta_{ij}\varepsilon_{kk} + \mu\left(\delta_{ik}\varepsilon_{kj} + \delta_{il}\varepsilon_{jl}\right) \\ &= \lambda\delta_{ij}\varepsilon_{kk} + \mu\left(\varepsilon_{ij} + \varepsilon_{ji}\right) = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}\end{aligned}$$

Demonstration with Excel

Examples

- In order for a material (with $\lambda = 115.384\ \mathrm{GPa}, \mu = 76.923\ \mathrm{GPa}$) to exhibit below elastic strain, what stress should be given?

$$\varepsilon = \begin{bmatrix} 0.002 & 0 & 0 \\ 0 & -0.0006 & 0 \\ 0 & 0 & -0.0006 \end{bmatrix}$$

- Hint: use "$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu\varepsilon_{ij}$"

```
200.000    0.000    0.000
  0.000    0.000    0.000
  0.000    0.000    0.000
```

Linear isotropic elasticity (Young, Poisson)

Elastic constitutive law (Hooke's law):

$$\varepsilon_{ij} = \frac{1}{E}\left[\sigma_{ij} - \nu\left(\sigma_{kk}\delta_{ij} - \sigma_{ij}\right)\right]$$

- If you apply below stress to a material (with $E = 200$ GPa, $\nu = 0.3$), in what strain tensor will the material exhibit?

$$\sigma = \begin{bmatrix} 200 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \text{the unit of stress is MPa}$$

```
0.00100  0.00000  0.00000
0.00000 -0.00030  0.00000
0.00000  0.00000 -0.00030
```

Notice that a material with $E = 200$ GPa, $\nu = 0.3$ behaves equivalently with a material with $\lambda = 115.384$ GPa, $\mu = 76.923$ GPa

Demonstration with Excel

Symmetries; why only two parameters?

- $\sigma_{ij} = \sigma_{ji}$ gives $\mathbb{E}_{ijkl} = \mathbb{E}_{jikl}$ thus, the required number of elastic constants reduces from 3x3x3x3 to 6x3x3.

- Similarly, $\varepsilon_{ij} = \varepsilon_{ji}$ gives $\mathbb{E}_{ijkl} = \mathbb{E}_{ijlk}$ so that we have the required of number of constants 6x6=36

The required number of constants can be further reduced. Consider the elastic energy:

$$\phi = \int \sigma_{ij} d\varepsilon_{ij}$$

$$\sigma_{ij} = \frac{\partial \phi}{\partial \varepsilon_{ij}} = \mathbb{E}_{ijkl}\varepsilon_{kl}$$

If we apply partial derivative once again, we have

$\frac{\partial^2 \phi}{\partial \varepsilon_{mn} \partial \varepsilon_{ij}} = \frac{\partial}{\partial \varepsilon_{mn}} \left(\mathbb{E}_{ijkl}\varepsilon_{kl} \right)$ since $\mathbb{E}$ is 'constant', we have

$$\frac{\partial^2 \phi}{\partial \varepsilon_{mn} \partial \varepsilon_{ij}} = \mathbb{E}_{ijkl} \left(\frac{\partial \varepsilon_{kl}}{\partial \varepsilon_{mn}} \right) = \mathbb{E}_{ijkl}\delta_{km}\delta_{ln} = \mathbb{E}_{ijmn}$$

Symmetries; why only two parameters?

- $\phi = \int \sigma_{ij} d\varepsilon_{ij}$
- $\sigma_{ij} = \frac{\partial \phi}{\partial \varepsilon_{ij}} = \mathbb{E}_{ijkl}\varepsilon_{kl}$
- If we apply partial derivative once again, we have
- $\frac{\partial^2 \phi}{\partial \varepsilon_{mn}\partial \varepsilon_{ij}} = \frac{\partial}{\partial \varepsilon_{mn}}\left(\mathbb{E}_{ijkl}\varepsilon_{kl}\right)$ since $\mathbb{E}$ is 'constant', we have
- $\frac{\partial^2 \phi}{\partial \varepsilon_{mn}\partial \varepsilon_{ij}} = \mathbb{E}_{ijkl}\left(\frac{\partial \varepsilon_{kl}}{\partial \varepsilon_{mn}}\right) = \mathbb{E}_{ijkl}\delta_{km}\delta_{ln} = \mathbb{E}_{ijmn}$

- We could do the 2nd order derivative in a different way (say, instead of $\frac{\partial^2 \phi}{\partial \varepsilon_{mn}\partial \varepsilon_{ij}}$ we could have done $\frac{\partial^2 \phi}{\partial \varepsilon_{ij}\partial \varepsilon_{mn}} = \frac{\partial}{\partial \varepsilon_{ij}}\left(\frac{\partial \phi}{\partial \varepsilon_{mn}}\right) = \frac{\partial}{\partial \varepsilon_{ij}}\left(\frac{\partial \phi}{\partial \varepsilon_{mn}}\right) = \mathbb{E}_{mnij}$
- The two cases (regardless of the order of derivative) should give equivalent result so that
- $\mathbb{E}_{ijmn} = \mathbb{E}_{mnij}$

- This summarizes our finding on the symmetries in elastic tensor:

Reduction to Voigt notation

- $\sigma_{21} = \mathbb{E}_{2111}\varepsilon_{11} + \mathbb{E}_{2112}\varepsilon_{12} + \mathbb{E}_{2113}\varepsilon_{13} + \mathbb{E}_{2121}\varepsilon_{21} + \mathbb{E}_{2122}\varepsilon_{22} + \mathbb{E}_{2123}\varepsilon_{23} + \mathbb{E}_{2131}\varepsilon_{31} + \mathbb{E}_{2132}\varepsilon_{32} + \mathbb{E}_{2123}\varepsilon_{33}$

- $$\sigma_{21} = \begin{bmatrix} \mathbb{E}_{2111} \\ \mathbb{E}_{2112} \\ \mathbb{E}_{2113} \\ \mathbb{E}_{2121} \\ \mathbb{E}_{2122} \\ \mathbb{E}_{2123} \\ \mathbb{E}_{2131} \\ \mathbb{E}_{2132} \\ \mathbb{E}_{2133} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{21} \\ \varepsilon_{22} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{32} \\ \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \mathbb{E}_{2111} \\ 2\mathbb{E}_{2112} \\ 2\mathbb{E}_{2113} \\ - \\ \mathbb{E}_{2122} \\ 2\mathbb{E}_{2123} \\ - \\ - \\ \mathbb{E}_{2133} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ - \\ \varepsilon_{22} \\ \varepsilon_{23} \\ - \\ - \\ \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \mathbb{E}_{2111} \\ \mathbb{E}_{2122} \\ \mathbb{E}_{2133} \\ 2\mathbb{E}_{2123} \\ 2\mathbb{E}_{2113} \\ 2\mathbb{E}_{2112} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{bmatrix}$$

- $$or \quad = \begin{bmatrix} \mathbb{E}_{2111} \\ \mathbb{E}_{2122} \\ \mathbb{E}_{2133} \\ \mathbb{E}_{2123} \\ \mathbb{E}_{2113} \\ \mathbb{E}_{2112} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix}$$ with $\gamma_{12} = 2\varepsilon_{12}$ and so forth

$$\sigma_{21} = \begin{bmatrix} \mathbb{E}_{21,1} \\ \mathbb{E}_{21,2} \\ \mathbb{E}_{21,3} \\ \mathbb{E}_{21,4} \\ \mathbb{E}_{21,5} \\ \mathbb{E}_{21,6} \end{bmatrix} \begin{bmatrix} \varepsilon_{1} \\ \varepsilon_{2} \\ \varepsilon_{3} \\ \gamma_{4} \\ \gamma_{5} \\ \gamma_{6} \end{bmatrix}$$

with $(1,1) \to (1), (2,2) \to (2), (3,3) \to (3)$
$(2,3) \to (4), (1,3) \to (5), (1,2) \to (6)$

Reduction to Voigt notation

$$\sigma_{21} = \begin{bmatrix} \mathbb{E}_{21,1} \\ \mathbb{E}_{21,2} \\ \mathbb{E}_{21,3} \\ \mathbb{E}_{21,4} \\ \mathbb{E}_{21,5} \\ \mathbb{E}_{21,6} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{bmatrix}$$

with $(1,1) \rightarrow (1), (2,2) \rightarrow (2), (3,3) \rightarrow (3)$
$(2,3) \rightarrow (4), (1,3) \rightarrow (5), (1,2) \rightarrow (6)$

$$\sigma_{21} = \begin{bmatrix} \mathbb{E}_{21,1} \\ \mathbb{E}_{21,2} \\ \mathbb{E}_{21,3} \\ \mathbb{E}_{21,4} \\ \mathbb{E}_{21,5} \\ \mathbb{E}_{21,6} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}$$

with $(1,2) \rightarrow (2,1) \rightarrow (6)$

$$\sigma_{6} = \begin{bmatrix} \mathbb{E}_{6,1} \\ \mathbb{E}_{6,2} \\ \mathbb{E}_{6,3} \\ \mathbb{E}_{6,4} \\ \mathbb{E}_{6,5} \\ \mathbb{E}_{6,6} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}$$

with $(1,2) \rightarrow (2,1) \rightarrow (6)$

Reduction to Voigt notation

$$\sigma_{ij} = \mathbb{E}_{ijkl}\varepsilon_{kl}$$

$$\sigma_i = \mathbb{E}_{ij}\varepsilon_j$$

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix}\mathbb{E}_{1111} & \mathbb{E}_{1122} & \mathbb{E}_{1133} & \mathbb{E}_{1123} & \mathbb{E}_{1113} & \mathbb{E}_{1112}\\ \mathbb{E}_{2211} & \mathbb{E}_{2222} & \mathbb{E}_{2233} & \mathbb{E}_{2223} & \mathbb{E}_{2213} & \mathbb{E}_{2212}\\ \mathbb{E}_{3311} & \mathbb{E}_{3322} & \mathbb{E}_{3333} & \mathbb{E}_{3323} & \mathbb{E}_{3313} & \mathbb{E}_{3312}\\ \mathbb{E}_{2311} & \mathbb{E}_{2322} & \mathbb{E}_{2333} & \mathbb{E}_{2323} & \mathbb{E}_{2313} & \mathbb{E}_{2312}\\ \mathbb{E}_{1311} & \mathbb{E}_{1322} & \mathbb{E}_{1333} & \mathbb{E}_{1323} & \mathbb{E}_{1313} & \mathbb{E}_{1312}\\ \mathbb{E}_{1211} & \mathbb{E}_{1222} & \mathbb{E}_{1233} & \mathbb{E}_{1223} & \mathbb{E}_{1213} & \mathbb{E}_{1212}\end{bmatrix} \begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{12}\end{bmatrix}$$

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{bmatrix} = \begin{bmatrix}\mathbb{E}_{11} & \mathbb{E}_{12} & \mathbb{E}_{13} & \mathbb{E}_{14} & \mathbb{E}_{15} & \mathbb{E}_{16}\\ \mathbb{E}_{21} & \mathbb{E}_{22} & \mathbb{E}_{23} & \mathbb{E}_{24} & \mathbb{E}_{25} & \mathbb{E}_{26}\\ \mathbb{E}_{31} & \mathbb{E}_{32} & \mathbb{E}_{33} & \mathbb{E}_{34} & \mathbb{E}_{35} & \mathbb{E}_{36}\\ \mathbb{E}_{41} & \mathbb{E}_{42} & \mathbb{E}_{43} & \mathbb{E}_{44} & \mathbb{E}_{45} & \mathbb{E}_{46}\\ \mathbb{E}_{51} & \mathbb{E}_{52} & \mathbb{E}_{53} & \mathbb{E}_{54} & \mathbb{E}_{55} & \mathbb{E}_{56}\\ \mathbb{E}_{61} & \mathbb{E}_{62} & \mathbb{E}_{63} & \mathbb{E}_{64} & \mathbb{E}_{65} & \mathbb{E}_{66}\end{bmatrix} \begin{bmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_3\\ \epsilon_4\\ \epsilon_5\\ \epsilon_6\end{bmatrix}$$

How many constants are required?

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix}\mathbb{E}_{1111} & \mathbb{E}_{1122} & \mathbb{E}_{1133} & \mathbb{E}_{1123} & \mathbb{E}_{1113} & \mathbb{E}_{1112}\\ \mathbb{E}_{2211} & \mathbb{E}_{2222} & \mathbb{E}_{2233} & \mathbb{E}_{2223} & \mathbb{E}_{2213} & \mathbb{E}_{2212}\\ \mathbb{E}_{3311} & \mathbb{E}_{3322} & \mathbb{E}_{3333} & \mathbb{E}_{3323} & \mathbb{E}_{3313} & \mathbb{E}_{3312}\\ \mathbb{E}_{2311} & \mathbb{E}_{2322} & \mathbb{E}_{2333} & \mathbb{E}_{2323} & \mathbb{E}_{2313} & \mathbb{E}_{2312}\\ \mathbb{E}_{1311} & \mathbb{E}_{1322} & \mathbb{E}_{1333} & \mathbb{E}_{1323} & \mathbb{E}_{1313} & \mathbb{E}_{1312}\\ \mathbb{E}_{1211} & \mathbb{E}_{1222} & \mathbb{E}_{1233} & \mathbb{E}_{1223} & \mathbb{E}_{1213} & \mathbb{E}_{1212}\end{bmatrix} \begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{12}\end{bmatrix}$$

How many constants do we need?

If the coordinate system happens to give strain and stress all principal values:

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\end{bmatrix} = \begin{bmatrix}\mathbb{E}_{1111} & \mathbb{E}_{1122} & \mathbb{E}_{1133}\\ & \mathbb{E}_{2222} & \mathbb{E}_{2223}\\ & & \mathbb{E}_{3333}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\end{bmatrix}$$

example

- Fe(1-0.025)-Al(0.025) alloy의 탄성 계수는 다음과 같이 주어진다.

- $\mathbb{E}_{11}$ = 270.71, $\mathbb{E}_{12}$ = 128.03, $\mathbb{E}_{44}$ = 108.77
- Fe-Al alloy는 Body-centered cubic 결정 구조를 가지고, 결정 대칭성에 의해 다음과 같은 탄성 거동을 한다.

$$\begin{bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\sigma_4\\\sigma_5\\\sigma_6\end{bmatrix}=\begin{bmatrix}\mathbb{E}_{11}&\mathbb{E}_{12}&\mathbb{E}_{13}&0&0&0\\\mathbb{E}_{21}&\mathbb{E}_{22}&\mathbb{E}_{23}&0&0&0\\\mathbb{E}_{31}&\mathbb{E}_{32}&\mathbb{E}_{33}&0&0&0\\0&0&0&\mathbb{E}_{44}&0&0\\0&0&0&0&\mathbb{E}_{55}&0\\0&0&0&0&0&\mathbb{E}_{66}\end{bmatrix}\begin{bmatrix}\epsilon_1\\\epsilon_2\\\epsilon_3\\\epsilon_4\\\epsilon_5\\\epsilon_6\end{bmatrix}$$

- 뿐만 아니라, cubic 결정구조의 대칭성으로 인해 $\mathbb{E}_{11} = \mathbb{E}_{22} = \mathbb{E}_{33}, \mathbb{E}_{44} = \mathbb{E}_{55} = \mathbb{E}_{66}, \mathbb{E}_{12} = \mathbb{E}_{13} = \mathbb{E}_{23}$

Example

- Fe(1-0.025)-Al(0.025) alloy의 단결정에 다음과 같은 탄성 변형률이 나타나기 위해 필요한 응력 상태는?

$$\begin{bmatrix} 0.0001 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Voigt notation

$$\begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{23}\\\sigma_{13}\\\sigma_{12}\end{bmatrix} = \begin{bmatrix}C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112}\\ & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212}\\ & & C_{3333} & C_{3323} & C_{3313} & C_{3312}\\ & & & C_{2323} & C_{2313} & C_{2312}\\ & symm & & & C_{1313} & C_{1312}\\ & & & & & C_{1212}\end{bmatrix} \begin{bmatrix}\epsilon_{11}\\\epsilon_{22}\\\epsilon_{33}\\2\epsilon_{23}\\2\epsilon_{13}\\2\epsilon_{12}\end{bmatrix}$$

$$\begin{bmatrix}\sigma_{1}\\\sigma_{2}\\\sigma_{3}\\\sigma_{4}\\\sigma_{5}\\\sigma_{6}\end{bmatrix} = \begin{bmatrix}C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ & & C_{33} & C_{34} & C_{35} & C_{36}\\ & & & C_{44} & C_{45} & C_{46}\\ & symm & & & C_{55} & C_{56}\\ & & & & & C_{66}\end{bmatrix} \begin{bmatrix}\epsilon_{1}\\\epsilon_{2}\\\epsilon_{3}\\\epsilon_{4}\\\epsilon_{5}\\\epsilon_{6}\end{bmatrix}$$

Cartesian <-> Voigt (cheat sheet)

```
c     stress, stiffness
      SUBROUTINE VOIGT(T1,T2,C2,C4,IOPT)
      IMPLICIT NONE
      REAL*8 T1(6),T2(3,3),C2(6,6),C4(3,3,3,3)
      INTEGER IJV(6,2),I,J,IOPT,I1,I2,J1,J2,N,M
      DATA ((IJV(N,M),M=1,2),N=1,6)/1,1,2,2,3,3,1,2,1,3,2,3/

      IF(IOPT.EQ.1) THEN
      DO I=1,6
         I1=IJV(I,1)
         I2=IJV(I,2)
         T2(I1,I2)=T1(I)
         T2(I2,I1)=T1(I)
      ENDDO
      ENDIF
C
      IF(IOPT.EQ.2) THEN
         DO I=1,6
            I1=IJV(I,1)
            I2=IJV(I,2)
            T1(I)=T2(I1,I2)
         ENDDO
         ENDIF
C
      IF (IOPT.EQ.3) THEN
         DO I=1,6
            I1=IJV(I,1)
            I2=IJV(I,2)
            DO J=1,6
               J1=IJV(J,1)
               J2=IJV(J,2)
               C4(I1,I2,J1,J2)=C2(I,J)
               C4(I2,I1,J1,J2)=C2(I,J)
               C4(I1,I2,J2,J1)=C2(I,J)
               C4(I2,I1,J2,J1)=C2(I,J)
            ENDDO
         ENDDO
      ENDIF
C
      IF(IOPT.EQ.4) THEN
         DO  I=1,6
            I1=IJV(I,1)
            I2=IJV(I,2)
            DO  J=1,6
               J1=IJV(J,1)
               J2=IJV(J,2)
               C2(I,J)=C4(I1,I2,J1,J2)
            ENDDO
         ENDDO
      ENDIF
C
      RETURN
      END
```

Convert from Cartesian to Voigt

- $\mathbb{E}_{11}^{(\text{voigt})} = \mathbb{E}_{1111}^{(\text{cartesian})}, \mathbb{E}_{23}^{(\text{voigt})} = \mathbb{E}_{2233}^{(\text{cartesian})}, \mathbb{E}_{41}^{(\text{voigt})} = \mathbb{E}_{2311}^{(\text{cartesian})}$

- Material anisotropy
- Symmetry can be represented by an orthogonal second order tensor,
- $\boldsymbol{Q} = Q_{ij}\mathbf{e}_\mathrm{i} \otimes \mathbf{e}_\mathrm{j}$, such that $\boldsymbol{Q}^{-1} = \boldsymbol{Q}^T$
- The invariance of the stiffness tensor under these transformations (due to symmetry) is:
- $\mathbb{E}^{(new)} = \boldsymbol{Q} \cdot \boldsymbol{Q} \cdot \mathbb{E}^{(old)} \cdot \boldsymbol{Q}^T \cdot \boldsymbol{Q}^T$ due to symmetry the resulting tensor should be equivalent with the original one: $\mathbb{E}^{(new)} \equiv \mathbb{E}^{(old)}$

http://web.mit.edu/16.20/homepage/3_Constitutive/Constitutive_files/module_3_with_solutions.pdf

Convert from Cartesian to Voigt

- $\mathbb{E}_{11}^{(\text{voigt})} = \mathbb{E}_{1111}^{(\text{cartesian})}, \mathbb{E}_{23}^{(\text{voigt})} = \mathbb{E}_{2233}^{(\text{cartesian})}, \mathbb{E}_{41}^{(\text{voigt})} = \mathbb{E}_{2311}^{(\text{cartesian})}$

- Material anisotropy
- Symmetry can be represented by an orthogonal second order tensor,
- $\boldsymbol{Q} = Q_{ij}\mathbf{e}_{\mathrm{i}} \otimes \mathbf{e}_{\mathrm{j}}$, such that $\boldsymbol{Q}^{-1} = \boldsymbol{Q}^{T}$
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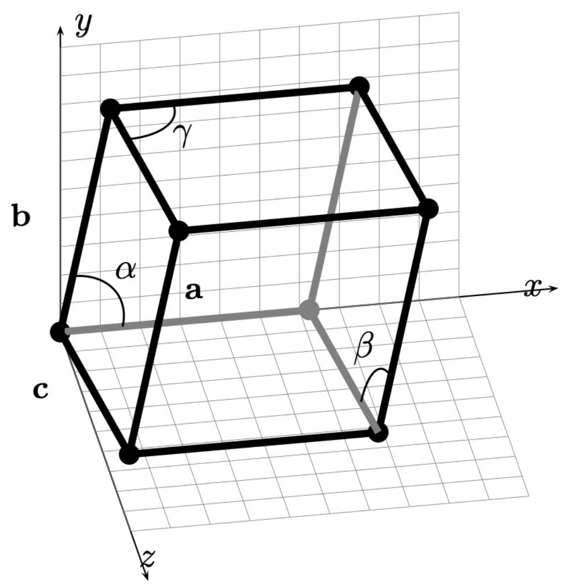
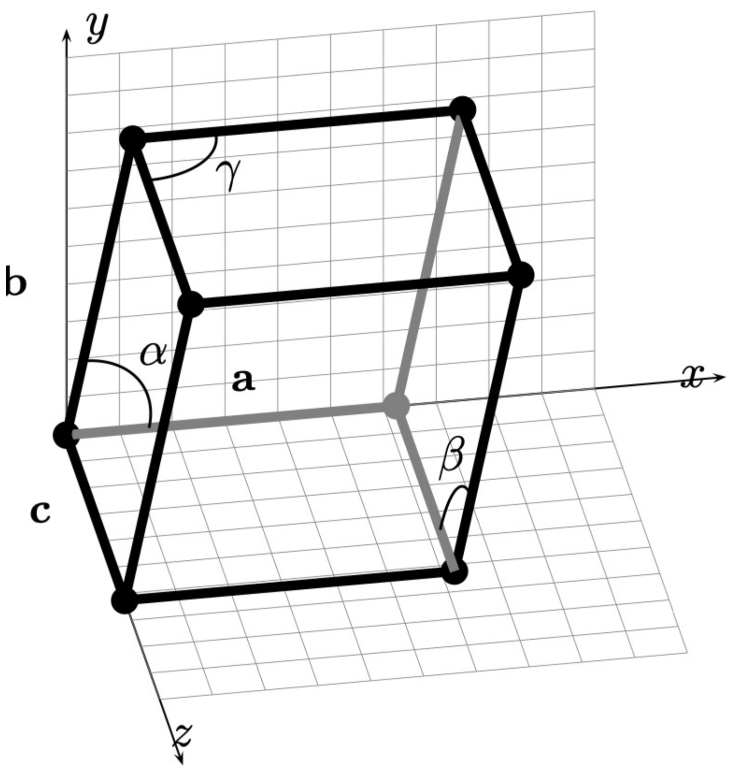
Triclinic (no symmetry)



Triclinic: no symmetry planes, fully anisotropic.
$\alpha, \beta, \gamma < 90$
Number of independent coefficients: 21
Symmetry transformation: None

$$C = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ & & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ & & & C_{2323} & C_{2313} & C_{2312} \\ & symm & & & C_{1313} & C_{1312} \\ & & & & & C_{1212} \end{bmatrix}$$

monoclinic (one symmetry plane)



Monoclinic: one symmetry plane (xy). $a \neq b \neq c,\ \beta = \gamma = 90, \alpha < 90$
Number of independent coefficients: 13
Symmetry transformation: reflection about z-axis

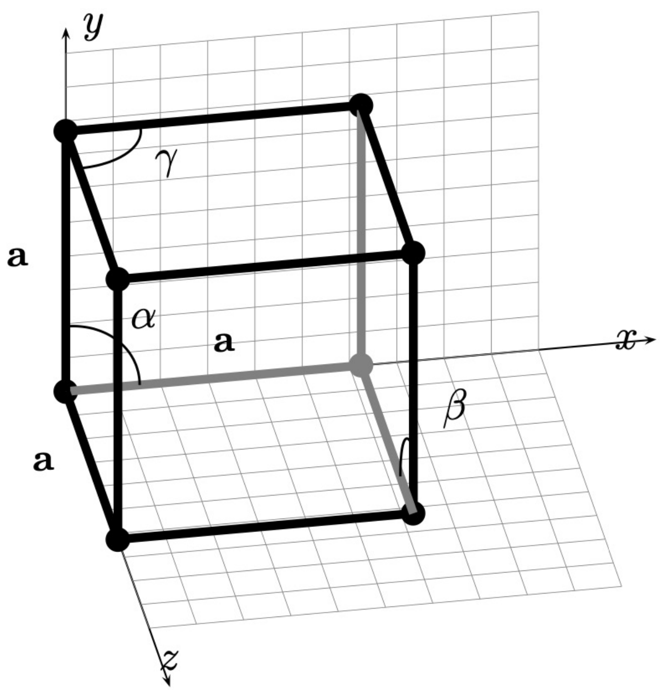
$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & C_{1112} \\ & C_{2222} & C_{2233} & 0 & 0 & C_{2212} \\ & & C_{3333} & 0 & 0 & C_{3312} \\ & & & C_{2323} & C_{2313} & 0 \\ & symm & & & C_{1313} & 0 \\ & & & & & C_{1212} \end{bmatrix}$$

monoclinic (one symmetry plane)

- For the case of Monoclinic:
- $\boldsymbol{Q} = Q_{ij}\mathbf{e}_\mathrm{i} \otimes \mathbf{e}_\mathrm{j} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
- Let's take a look at the invariance due to symmetry
- $\mathbb{E}^{(new)} = \boldsymbol{Q} \cdot \boldsymbol{Q} \cdot \mathbb{E}^{(old)} \cdot \boldsymbol{Q}^T \cdot \boldsymbol{Q}^T$ due to symmetry the resulting tensor should be equivalent with the original one: $\mathbb{E}^{(new)} \equiv \mathbb{E}^{(old)}$
- In its matrix form:
  - $\mathbb{E}^{(new)}_{ijkl} = Q_{im}Q_{jn}Q_{ko}Q_{lp}\mathbb{E}^{(old)}_{mnop}$
  - Ex: $\mathbb{E}^{(voigt)}_{11} = \mathbb{E}_{1111} = Q_{1m}Q_{1n}Q_{1o}Q_{1p}\mathbb{E}^{(old)}_{mnop}$
    - If you look at the matrix form of symmetry operator Q in the above, only diagonal components are non-zero. Therefore, $Q_{ij} = 0$ if $i \neq j$.
    - $\mathbb{E}^{(voigt)}_{11} = \mathbb{E}_{1111} = Q_{11}Q_{11}Q_{11}Q_{11}\mathbb{E}^{(old)}_{1111} = \mathbb{E}^{(old)}_{1111}$
    - Therefore, $\mathbb{E}^{(voigt)}_{11} = \mathbb{E}_{1111}$
  - Ex: $\mathbb{E}^{(voigt)}_{14} = \mathbb{E}_{1123} = Q_{1m}Q_{1n}Q_{2o}Q_{3p}\mathbb{E}^{(old)}_{mnop} = Q_{11}Q_{11}Q_{22}Q_{33}\mathbb{E}^{(old)}_{1123} = 1\times1\times1\times(-1)\times\mathbb{E}^{(old)}_{1123} = -\mathbb{E}^{(old)}_{1123}$
    - Therefore, in order to satisfy $\mathbb{E}_{1123} = -\mathbb{E}_{1123}$, $\mathbb{E}_{1123}$ should be zero.

Cubic



Cubic: three mutually orthogonal planes of reflection symmetry plus 90° rotation symmetry with respect to those planes. $a = b = c,\ \alpha = \beta = \gamma = 90$
Number of independent coefficients: 3
Symmetry transformations: reflections and 90° rotations about all three orthogonal planes

$$Q = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} C_{1111} & C_{1122} & C_{1122} & 0 & 0 & 0 \\ & C_{1111} & C_{1122} & 0 & 0 & 0 \\ & & C_{1111} & 0 & 0 & 0 \\ & & & C_{1212} & 0 & 0 \\ & symm & & & C_{1212} & 0 \\ & & & & & C_{1212} \end{bmatrix}$$