Vectors and Matrices operations required to study metal plasticity

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Nomenclature

- Rule(1) 굵은 글씨체(bold face)로 쓰여진 알파벳 기호 (가령 $\boldsymbol{b}$) 는 그 기호가 가르키는 '물리량'이 벡터임을 의미한다.

- Rule(2) 벡터$\boldsymbol{b}$를 구성성분을 사용하여 표기할 수도 있다. $\boldsymbol{b} = (b_1, b_2, b_3)$. 각 구성성분($b_i$ with $i = 1,2,3$)이 굵은 글씨체가 아닌 글씨체로 쓰여져 있음을 확인하라. (왜?)

- Rule(3) 굵은 글씨체로 쓰인 대문자 $\boldsymbol{A}$ 는 2nd order tensor (혹은 3x3 matrix)를 나타내는데 다음과 같이 사용될 수 있다.

Nomenclature

- In matrix notation, the subscripts (also called as indices) are used to denote the column and the row of the associated components.
- Say, A_{ij} refers to the component in i-th row (행) and j-th column (렬)[+].
- Example: $\boldsymbol{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$
- We preferably use Cartesian coordinates consisting of three basis vectors (often denoted as $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ or equivalently as $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$)[*]

[+](Mnemonic) 행과 열을 구분하여 외우는 팁: 가로세로(o), 세로가로(x); 행렬. 따라서 행은 가로, 열은 세로. Row와 column중 column은 '기둥'을 뜻하고 기둥은 세워져 있다. 따라서 Column은 행, 나머지 row는 열.

[*]The basis vectors are written in **bold-face**, implying that they are **vectors** <u>not scalars</u>.

Why do we study vectors, tensors, coordinate systems?

- 재료는 근원적으로 3차원이며, 스칼라 물리량으로 재료의 거동을 설명하기에 부족하다.
- 재료의 거동에서 이방성(anisotropy)가 높아, 방향마다 재료의 거동이 달라질 수 있기 때문이다. (e.g., Miller index)
- 스칼라만을 활용한 물리모형들의 활용도가 매우 제한적이며, 재료가 가진 이방성을 설명하거나 3차원 공간에서의 재료의 역학 거동을 설명할 수 없다. (한방향으로의 길이만 가진 1D 재료는 없다! 따라서 예를 들어 scalar 변형률로 재료구성방정식 사용불가)

벡터 (vector)

- A vector in 2-dimensional space has two independent components.
- Say, a vector $\boldsymbol{a}$ has two separate components $\boldsymbol{a} = (a_1, a_2)$. For example, vector $\boldsymbol{b} = (2,3)$
- In 3D, a vector has three components.
- Length (magnitude) of vector

$$|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{\sum_i^3 a_i^2}$$

- vector $\boldsymbol{a}$의 단위 벡터는 다음과 같다.

$$\frac{\boldsymbol{a}}{|\boldsymbol{a}|} = \left(\frac{a_1}{\sqrt{a_1^2 + a_2^2 + a_3^2}}, \frac{a_2}{\sqrt{a_1^2 + a_2^2 + a_3^2}}, \frac{a_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}}\right)$$

예제)

Ex1) 다음 벡터의 length (magnitude)를 구하시오.

$$\boldsymbol{a} = (1,2,5)$$

Ex2) 다음 벡터의 unit vector를 구하시오.

$$\boldsymbol{b} = (1,1,2)$$

벡터 (vector) operations

- Vector addition: addition of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be expressed as:

either $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$ or $c_i = a_i + b_i$ with $i = 1,2,3$

- The former notation is called vector notation (using bold-face)
- The latter notation has many names and is usually called Einstein notation;
- In order to perform vector addition for each component, make sure that the associated vectors (tensors) are referred to the same coordinate system.

벡터 (vector)

- Vector multiplication with scalar

$$c\boldsymbol{a} = (ca_1, ca_2, ca_3) = c(a_1, a_2, a_3)$$

- A vector decomposed into three vectors aligned with basis vectors of given coordinates:

$$\boldsymbol{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$$

Some people use $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ to denote the basis vectors such that

$$\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$$

예제)

Ex1) 다음 두 벡터 합을 구하시오.

$$\boldsymbol{a} = (1,2,5) \quad \boldsymbol{b} = (2,-2,0)$$

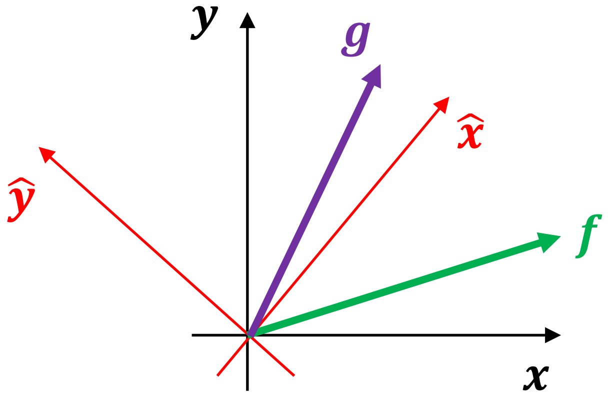
Ex2) 다음 두 벡터의 합에 해당하는 벡터의 unit vector를 구하시오.

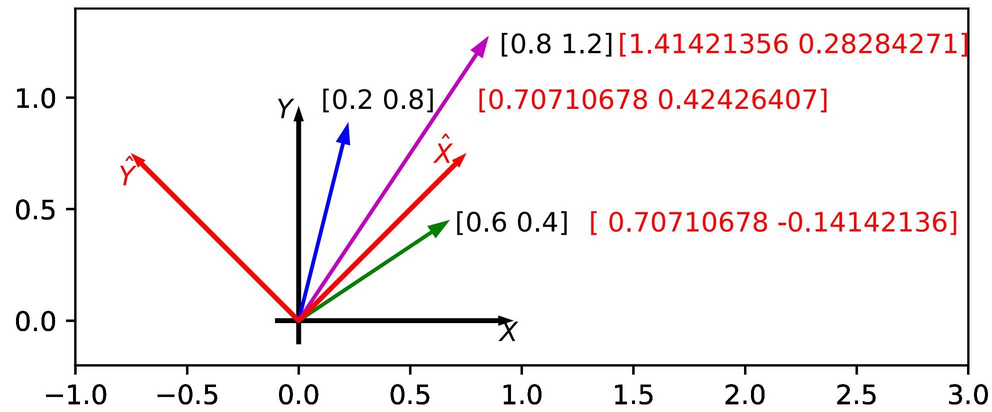
$$\boldsymbol{a} = (-1,3,0) \quad \boldsymbol{b} = (2,-2,1)$$

Vector operations and coordinates

- 벡터의 구성성분은 임의로 설정된 좌표계에 의해 특정된다. 합하는 두 벡터에 사용하는 좌표계가 다르다면, 같은 물리량 (예를 들어 힘; force)을 표현하는 벡터라도 다른 좌표 값을 가진다.
- 힘 벡터의 합을 구성성분의 합으로 표현하기 위해서는 **반드시** 두 벡터의 구성성분이 같은 좌표계에 표현되어 있어야 한다.

각각 특정 물리량을 표현하는 ***g***와 ***f***의 합은 어떠한 좌표계를 사용하던 그 '물리적' 결과가 동일해야 한다.





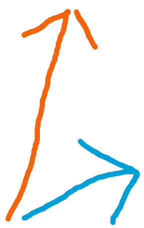
- 합 뿐만 아니라 다른 벡터 operations들을 행하기에 앞서 반드시 구성성분이 같은 좌표계로 표현되어 있어야 한다.
- Operation 결과가 vector (혹은 tensor) quantity 일때, 구성성분들은 선택된 좌표계에 참조된다.

The scalar product

FYI, $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ (Kronecker delta)

$$\delta_{ij} = 1\ (if\ i = j);\ \delta_{ij} = 0\ (i \neq j)$$

Geometric representation of scalar product of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$



$$\boldsymbol{x} \cdot \boldsymbol{y} = |\boldsymbol{x}||\boldsymbol{y}| \cos\theta$$

$\cos\theta$ is an even function, so that

$$\boldsymbol{x} \cdot \boldsymbol{y} = |\boldsymbol{x}||\boldsymbol{y}| \cos(-\theta) = |\boldsymbol{y}||\boldsymbol{x}| \cos(\theta) = \boldsymbol{y} \cdot \boldsymbol{x}$$

$$|\boldsymbol{x}| \equiv x = (\boldsymbol{x} \cdot \boldsymbol{x})^{1/2}$$

Algebraic representation of scalar product of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$

$$\boldsymbol{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = \sum_i x_i\mathbf{e}_i$$

$$\boldsymbol{y} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = \sum_i y_i\mathbf{e}_\mathrm{i}$$

$$\boldsymbol{x} \cdot \boldsymbol{y} = \left(\sum_i x_i\mathbf{e}_i\right) \cdot \left(\sum_i y_j\mathbf{e}_j\right)$$

$$= \sum_i\sum_j x_i y_j \mathbf{e}_i \cdot \mathbf{e}_j = \sum_i\sum_j x_i y_j \delta_{ij}$$

$$= \sum_i\sum_j x_i y_j \delta_{ij} = \sum_i x_i y_i$$

벡터 (vector) operations

- Dot product aka inner dot product (내적):

$d = \boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$ or $d = \sum_i^3 a_i b_i \rightarrow$ (Einstein): $d = a_i b_i$

- Alternative form:

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta$$

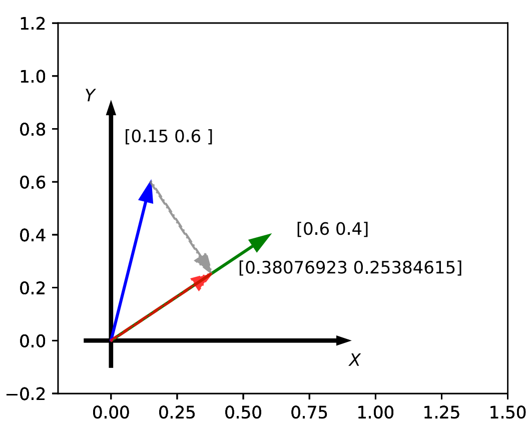
$\boldsymbol{\theta}$ denotes the angle between the two vectors ($\boldsymbol{a}$ and $\boldsymbol{b}$).

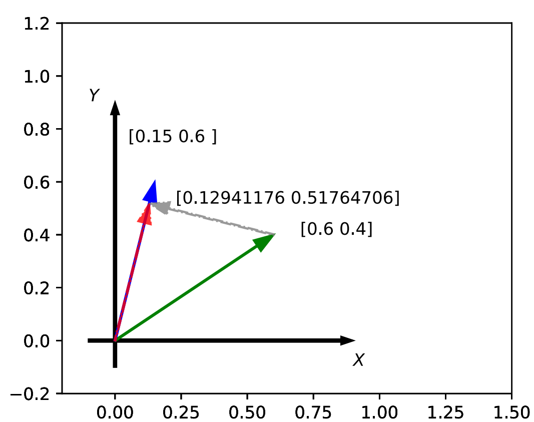
$$\boldsymbol{a} \cdot \boldsymbol{b} = \left(a_x \boldsymbol{i} + a_y \boldsymbol{j} + a_z \boldsymbol{k}\right) \cdot \left(b_x \boldsymbol{i} + b_y \boldsymbol{j} + b_z \boldsymbol{k}\right)$$

where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are unit vectors along the axes *x,y,z, respectively.*

- Inner product of different basis vector leads in zero, while that of the same basis vectors lead to 1: $\boldsymbol{i} \cdot \boldsymbol{j} = 0$ *and* $\boldsymbol{i} \cdot \boldsymbol{i} = 1$

$\rightarrow$ $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ (Kronecker delta)





Either way, the dot product amounts to ~42.27

예제)

- 다음은 Miller index로 나타낸 BCC 결정 구조내의 면과 방향이다. 두 방향사이의 끼인각은?

$$\boldsymbol{n} = (1,1,0)$$
$$\boldsymbol{b} = [1, \bar{1}, 0]$$

- 다음 결정면 $\boldsymbol{n}$ 과 방향 $\boldsymbol{b}$으로 이루어진 slip system이 FCC 결정내 존재할까 ?

$$\boldsymbol{n} = (1, \bar{1}, 1)$$
$$\boldsymbol{b} = [1, \bar{1}, 0]$$

Einstein summation convention

- Albert Einstein이 벡터와 텐서등의 물리량을 이용하여 그의 이론을 논문으로 쓰면서 (그의 물리법칙과는 무관한) 재미있는 사실을 하나 관찰했다. 벡터나 텐서가 inner dot, cross product, 등등의 operations 참여하면서 '덧셈'이 있을시에 반드시 해당 subscript가 두번씩 나타나고, 반대로 subscript가 두번씩 반복되어 나타나면 '덧셈'이 존재한다는 것이다. 따라서, 언제나 두개의 동일한 subscript가 나타나면 간단히 summation 기호를 없애도 된다고 생각했다. 예를 들면 $x_i y_i$와 같은 표현이 수식에 나오면 이것은 따로 말을 하지 않더라도 $\sum_i^n x_i y_i$ 을 의미한다는 사실이다 (이때 n은 물리량이 표혀된 공간의 차원이다). 유사하게, 만약 두쌍의 subscript가 반복된다면, 두개의 summation 기호가 생략된다.

Examples of Einstein summation

- $\boldsymbol{x} = x_i \mathbf{e}_i = \sum_{i}^{n} x_i \mathbf{e}_i = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + \cdots + x_n \mathbf{e}_n$
- $\boldsymbol{x} \cdot \boldsymbol{y} = x_i y_j \delta_{ij} = x_i y_i = x_j y_j = x_1 y_1 + x_2 y_2 + \cdots x_n y_n$
- $\boldsymbol{x} \cdot \mathbf{e}_i = x_j \mathbf{e}_j \cdot \mathbf{e}_i = x_j \delta_{ij} = x_i \begin{cases} \boldsymbol{x} \cdot \mathbf{e}_1 = x_1 \\ \boldsymbol{x} \cdot \mathbf{e}_2 = x_2 \\ \boldsymbol{x} \cdot \mathbf{e}_3 = x_3 \end{cases}$

The last equation defines the components of vector. The same can be referred to as 'projection' of $\boldsymbol{x}$ on the $\mathbf{e}_i$ axis. ($\boldsymbol{x}$벡터의 $\mathbf{e}_i$축으로의 내적)

Ex)

- 3차원 공간에서의 물리량으로 이루어진 다음 expression을 Einstein summation convention을 사용하여 나타내시오.

$$\boldsymbol{b} = \boldsymbol{x} + \boldsymbol{C} \cdot \boldsymbol{y}$$

$\boldsymbol{C} \cdot \boldsymbol{y}$ 은 내적이며 $\boldsymbol{C}$가 2nd order tensor (3x3 matrix) 이고 $\boldsymbol{y}$는 벡터이다. 따라서 그 결과는

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} C_{11}y_1 + C_{12}y_2 + C_{13}y_3 \\ C_{21}y_2 + C_{22}y_2 + C_{23}y_3 \\ C_{31}y_1 + C_{32}y_2 + C_{33}y_3 \end{bmatrix} = \begin{bmatrix} \sum_j^3 C_{1j}y_j \\ \sum_j^3 C_{2j}y_j \\ \sum_j^3 C_{3j}y_j \end{bmatrix}$$

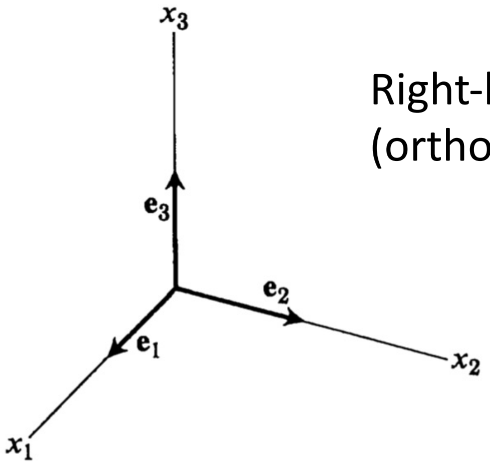
$$\boldsymbol{b} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \sum_j^3 C_{1j}y_j \\ \sum_j^3 C_{2j}y_j \\ \sum_j^3 C_{3j}y_j \end{bmatrix} \rightarrow \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \sum_j^3 C_{1j}y_j \\ \sum_j^3 C_{2j}y_j \\ \sum_j^3 C_{3j}y_j \end{bmatrix}$$

$$b_i = x_i + \sum_j^3 C_{ij}y_j \quad for\ i = 1,2,3$$

$$\rightarrow b_i = x_i + C_{ij}y_j$$

Cartesian coordinate system

- We confine our study to the cases using an ***orthonormal*** basis – three basis vectors are perpendicular each other, all of which has the length of unity.
- Now, we denote these three orthonormal basis vectors as $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$.



Right-handed Cartesian (orthonormal) coordinate system.

A vector $\boldsymbol{x}$ then can be expressed a linear combination of the three basis vectors such that

$$\boldsymbol{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$$

벡터 (vector) operations

- Dyadic product (a.k.a. outer product):

$$\boldsymbol{a} \otimes \boldsymbol{b} = \left(a_x \boldsymbol{i} + a_y \boldsymbol{j} + a_z \boldsymbol{k}\right) \otimes \left(b_x \boldsymbol{i} + b_y \boldsymbol{j} + b_z \boldsymbol{k}\right)$$

where $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are unit vectors along the axes *x,y,z, respectively.*

$$\begin{aligned}\boldsymbol{a} \otimes \boldsymbol{b} = {} & a_x b_x (\boldsymbol{i} \otimes \boldsymbol{i}) + a_x b_y (\boldsymbol{i} \otimes \boldsymbol{j}) + a_x b_z (\boldsymbol{i} \otimes \boldsymbol{k}) \\ & + a_y b_x (\boldsymbol{j} \otimes \boldsymbol{i}) + a_y b_y (\boldsymbol{j} \otimes \boldsymbol{j}) + a_y b_z (\boldsymbol{j} \otimes \boldsymbol{k}) \\ & + a_z b_x (\boldsymbol{k} \otimes \boldsymbol{i}) + a_z b_y (\boldsymbol{k} \otimes \boldsymbol{j}) + a_z b_z (\boldsymbol{k} \otimes \boldsymbol{k})\end{aligned}$$

Also equivalently,

$$\boldsymbol{a} \otimes \boldsymbol{b} = \begin{bmatrix} a_x b_x & a_x b_y & a_x b_z \\ a_y b_x & a_y b_y & a_y b_z \\ a_z b_x & a_z b_y & a_z b_z \end{bmatrix}$$

If $\boldsymbol{n}^s$ and $\boldsymbol{b}^s$ are slip system s (unit) plane normal and (unit) slip direction vectors, $\boldsymbol{n}^s \otimes \boldsymbol{b}^s$ *corresponds to Schmid tensor such that* $\boldsymbol{M}^s = \boldsymbol{n}^s \otimes \boldsymbol{b}^s$ *or* $\boldsymbol{M}^s_{ij} = \boldsymbol{n}^s_i \otimes \boldsymbol{b}^s_j$

Schmid tensor and resolved shear stress

- $\boldsymbol{n}^s = \frac{(1,1,1)}{|(1,1,1)|}$ and $\boldsymbol{b}^s = \frac{(1,0,-1)}{|(1,0,-1)|}$

Say, the crystal is subjected to stress tensor of

$$\boldsymbol{\sigma} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The resolved shear stress (RSS) amounts to

$$\tau^s = \boldsymbol{\sigma} \cdot \boldsymbol{n}^s \cdot \boldsymbol{b}^s = \boldsymbol{\sigma} : \boldsymbol{M}^s = \sigma_{ij} M_{ij}^s$$

$$\tau^s = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{6}} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Recall the Schmid law**:** $\tau^s = \sigma \cos\phi \cos\lambda$

** Caution, direct use of miller index for crystal plane normal and direction should be careful.
Crystal coordinate system of cubic (FCC, BCC) are equivalent to Cartesian. Less symmetric
Structures (such as triclinic) would require change of the miller indices to relevant components in Cartesian coordinates.

Pressure independence of slip

- $\boldsymbol{n}^s = \frac{(1,1,1)}{|(1,1,1)|}$ and $\boldsymbol{b}^s = \frac{(1,0,-1)}{|(1,0,-1)|}$

Say, the crystal is subjected to stress tensor of

$$\boldsymbol{\sigma} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \boldsymbol{\sigma} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \boldsymbol{\sigma} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Calculate the resolved shear stress for each stress tensor above, and discuss what you observed.

Identity matrix

- $\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- In the tensor notation, one would use the Kronecker delta denoted as δ_{ij}

$\delta_{ij} = 1 \text{ if } i = j$

$\delta_{ij} = 0 \text{ if } i \neq j$

Transpose

- $\boldsymbol{A} = \begin{bmatrix} 3 & 4 & 6 \\ -3 & 2 & 5 \\ 1 & -1 & -4 \end{bmatrix}$

- $\boldsymbol{A}^T = \begin{bmatrix} 3 & -3 & 1 \\ 4 & 2 & -1 \\ 6 & 5 & -4 \end{bmatrix}$

- In tensor notation, $A_{ij}^T = A_{ji}$

Matrix addition and multiplication

- Addition
  - $\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$
  - $C_{ij} = A_{ij} + B_{ij}$
- Multiplication (dot products)
  - $\boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{B}$
  - $C_{ij} = A_{ik}B_{kj}$
  - Multiplication is not commutative
    - $\boldsymbol{A} \cdot \boldsymbol{B} \neq \boldsymbol{B} \cdot \boldsymbol{A}$
- Double dot products
  - $d = \boldsymbol{A}\colon\boldsymbol{B}$ (denote d is a scalar quantity thus is **not** denoted in bold-face)
  - $d = A_{ij}B_{ij}$

Change basis (not necessarily orthonormal);

related with deformation gradient tensor

$\boldsymbol{v}^{new} = \boldsymbol{A} \cdot \boldsymbol{v}^{old}$ $\qquad v_i^{new} = A_{ij} v_j^{old}$

$\boldsymbol{A}$ has many different names: change-of-basis matrix. It changes from old basis vectors to new basis vectors.

Say, below matrix

$$\hat{\mathrm{e}}_i = A_{ij} \mathrm{e}_j$$

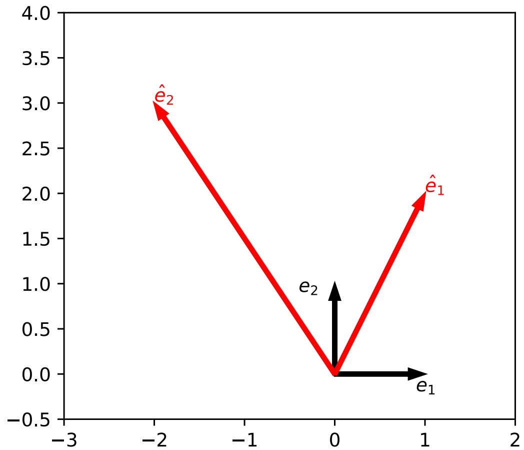
changes from basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$

Q) How to obtain A_{ij}?

A) $A_{ij} = \hat{\mathbf{e}}_i \cdot \mathbf{e}_j$

Derivation:

1) Use $\hat{\mathrm{e}}_i = A_{ij} \mathrm{e}_j$
2) Apply dot product with $\mathbf{e}_\mathrm{j}$



Left change in basis gives

$$A_{ij} = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$$

Check:

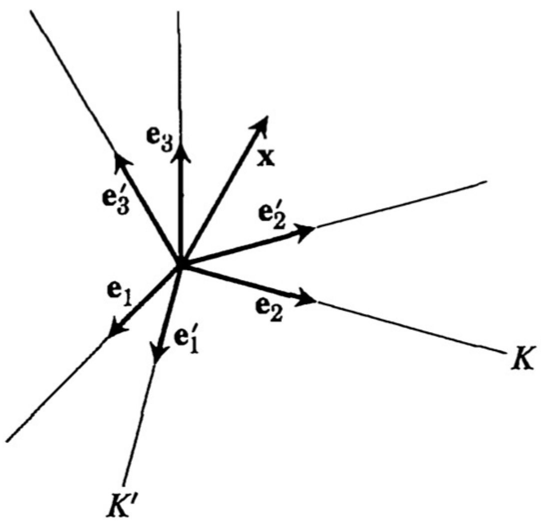
$$\boldsymbol{A} \cdot \mathbf{e}_1 = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \hat{\mathbf{e}}_1$$

$$\boldsymbol{A} \cdot \mathbf{e}_2 = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \hat{\mathbf{e}}_2$$

https://youtu.be/P2LTAUO1TdA

Rotation (transformation) of the coordinate system

Relationship between the components of a unit vector expressed with respect to two different Cartesian bases with the same origin (not necessarily orthonormal);



Two cartesian coordinates (K and K') with two separate sets of basis vectors ($\mathbf{e}_i$ and $\mathbf{e}'_i$) and a vector $\mathbf{x}$

Any vector $\mathbf{x}$ can be resolved into components with respect to either the K or the K' system.

$$\mathbf{x} = (\mathbf{x} \cdot \mathbf{e}_j)\mathbf{e}_j = x_j \mathbf{e}_j$$

If we take $\mathbf{x} = \mathbf{e}'_i$ (a certain basis vector of K')

$$\mathbf{e}'_i = (\mathbf{e}'_i \cdot \mathbf{e}_j)\mathbf{e}_j \equiv a_{ij}\mathbf{e}_j$$

The nine terms a_{ij} (for each of three basis vectors; $i = 1$, $i = 2$, and $i = 3$) are directional cosines of the angles between the six axes:

$$\boldsymbol{R} \equiv (a_{ij}) \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$\boldsymbol{R}$ is known as the transformation matrix (or rotation matrix) in three dimension.

Rotation (transformation) of the coordinate system

$\mathbf{e}'_i \equiv a_{ij}\mathbf{e}_j$ Switching $j \rightarrow k$ $\mathbf{e}'_i \equiv a_{ik}\mathbf{e}_k$

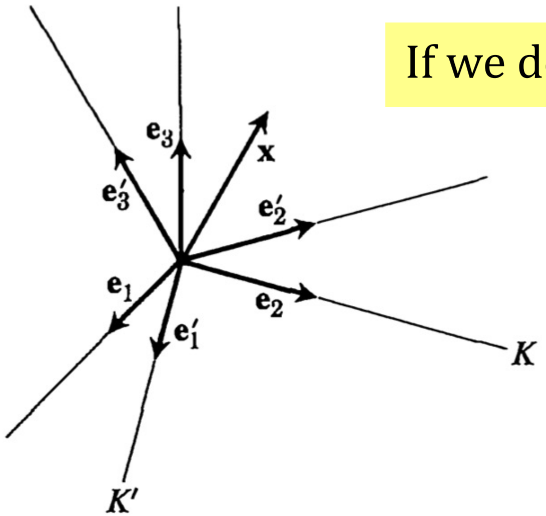
Earlier, we defined: $a_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j$

And $a_{ij}\mathbf{e}_j = \mathbf{e}'_i \cdot \mathbf{e}_j \cdot \mathbf{e}_j$

$\therefore a_{ij}\ \mathbf{e}_j = \mathbf{e}'_i|\mathbf{e}| \rightarrow a_{ij}\mathbf{e}_j = \mathbf{e}'_i$

$\delta_{ij} = \mathbf{e}'_i \cdot \mathbf{e}'_j = a_{ik}\mathbf{e}_k \cdot \mathbf{e}'_j = a_{ik}b_{kj}$ $\quad [a] = [b]^{-1}$

If we defined: $b_{ij} = \mathbf{e}_i \cdot \mathbf{e}'_j$



Two cartesian coordinates (K and K') with two separate sets of basis vectors ($\mathbf{e}_i$ and $\mathbf{e}'_i$) and a vector $\mathbf{x}$

Any vector $\mathbf{x}$ may be expressed in the K system as

$$\mathbf{x} = x_j\mathbf{e}_j$$

or as in the K' system using primed basis such as

$$\mathbf{x} = x'_i\mathbf{e}'_i$$

They are the same vector so one can equate

$$\mathbf{x} = x'_i\mathbf{e}'_i = x_j\mathbf{e}_j$$

One could replace $\mathbf{e}'_i$ with $a_{ij}\mathbf{e}_j$

$$x'_i a_{ij}\mathbf{e}_j = x_j\mathbf{e}_j$$

$$x_j = a_{ji}x'_i$$

so that

$$x_j = x'_i a_{ij} \quad [x_j]^T = [x'_i a_{ij}]^T \quad x_j = [a_{ij}]^T [x'_i]^T$$

Or equivalently, swapping the indices i and j gives:

$$x_i = a_{ij}x'_j$$

Inverse transformation?

$$\mathbf{e}_i = \mathbf{e}_i \times 1 = \mathbf{e}_i(\mathbf{e}'_j \cdot \mathbf{e}'_j) = (\mathbf{e}_i \cdot \mathbf{e}'_j)\mathbf{e}'_j = a_{ij}\mathbf{e}'_j$$

$$a_{kj}x_j = a_{kj}b_{jl}x'_l = \delta_{kl}x'_l = x'_k$$

In summary we have:

$$\mathbf{x} = x'_i\mathbf{e}'_i = x_j\mathbf{e}_j$$

$$\mathbf{e}'_i = a_{ij}\mathbf{e}_j, \qquad \mathbf{e}_i = a_{ji}\mathbf{e}'_j$$

$$x'_i = a_{ij}x_j, \qquad x_i = a_{ji}x'_j$$

$$a_{ik}a_{jk} = a_{ki}a_{kj} = \delta_{ij}$$

Earlier, we defined:

$$a_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j$$

$$b_{ij} = \mathbf{e}_i \cdot \mathbf{e}'_j$$

Earlier, we defined:

$$a_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j = \mathbf{e}_j \cdot \mathbf{e}'_i = b_{ji}$$

If the inner dot product of a and b matrices:

$$a_{ik}b_{kj} = (\mathbf{e}'_i \cdot \mathbf{e}_k)(\mathbf{e}_k \cdot \mathbf{e}'_j)$$

$$= \mathbf{e}'_i \cdot (\mathbf{e}_k \cdot \mathbf{e}_k) \cdot \mathbf{e}'_j$$

$$= \mathbf{e}'_i \cdot \mathbf{e}'_j = \delta_{ij}$$

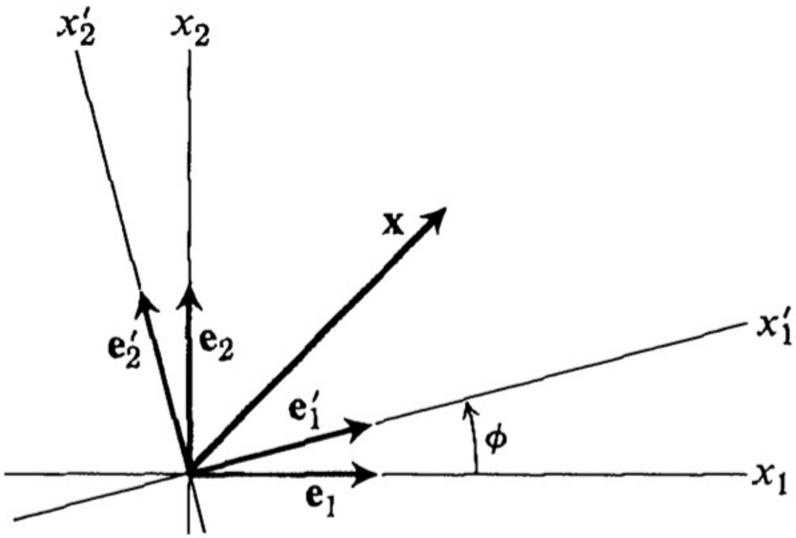
Scalar product is invariant under orthogonal transformations

$$\boldsymbol{x}' \cdot \boldsymbol{y}' = x_i' y_i' \quad = a_{ij} x_j a_{ik} y_k \quad = a_{ij} a_{ik} x_j y_k$$

$$= \delta_{jk} x_j y_k \ = x_j y_j \ = \boldsymbol{x} \cdot \boldsymbol{y}$$

$$a_{ij} a_{ik} = \left(a_{ji}\right)^T a_{ik} = b_{ji} a_{ik} = \delta_{jk}$$

Two dimensional case



$$a_{ij} \equiv (\mathbf{e}_i' \cdot \mathbf{e}_j), \qquad \text{for } i, j = 1{,}2$$

https://en.wikipedia.org/wiki/List_of_trigonometric_identities

Shift by one quarter period
$\sin(\theta \pm \frac{\pi}{2}) = \pm \cos\theta$
$\cos(\theta \pm \frac{\pi}{2}) = \mp \sin\theta$
$\tan(\theta \pm \frac{\pi}{4}) = \frac{\tan\theta \pm 1}{1 \mp \tan\theta}$
$\csc(\theta \pm \frac{\pi}{2}) = \pm \sec\theta$
$\sec(\theta \pm \frac{\pi}{2}) = \mp \csc\theta$
$\cot(\theta \pm \frac{\pi}{4}) = \frac{\cot\theta \pm 1}{1 \mp \cot\theta}$

$$[a_{ij}] = \begin{bmatrix} \cos\phi & \cos(90° - \phi) \\ \cos(90° + \phi) & \cos\phi \end{bmatrix}$$

Physical theories must be invariant to the choice of coordinate system

If we fix our attention on a physical vector (e.g. velocity) and then rotate the coordinate system $(K \rightarrow K')$, the vector will have different numerical components in the rotated coordinate system (as evident in the coordinate transformation rule we just discussed earlier). So we are led to realize that a vector is more than an ordered triple. Rather, it is many sets of ordered triples, which are related in a definite way. One still specifies a vector by giving three ordered numbers (components), but these three numbers are distinguished from an arbitrary collection of three numbers by including the law of coordinate transformation under rotation of the coordinate frame as part of the definition.

Thus, one physical vector may be represented by infinitely many sets of ordered triples. The particular triple depends on the chosen coordinate system of the observer.

This is important because physical laws (and results) must be the same regardless of coordinate system, that is, regardless of the orientation of observer's coordinate system.

Physical laws and coordinate system

- The importance of thinking of these quantities in terms of their transformation properties lies in the requirement that physical theories must be invariant under the change of the coordinate system.
- Physical laws should not be affected by the choice of a coordinate system.
- We'll examine this using an example in what follows.

Newton's second law

Algebraic representation

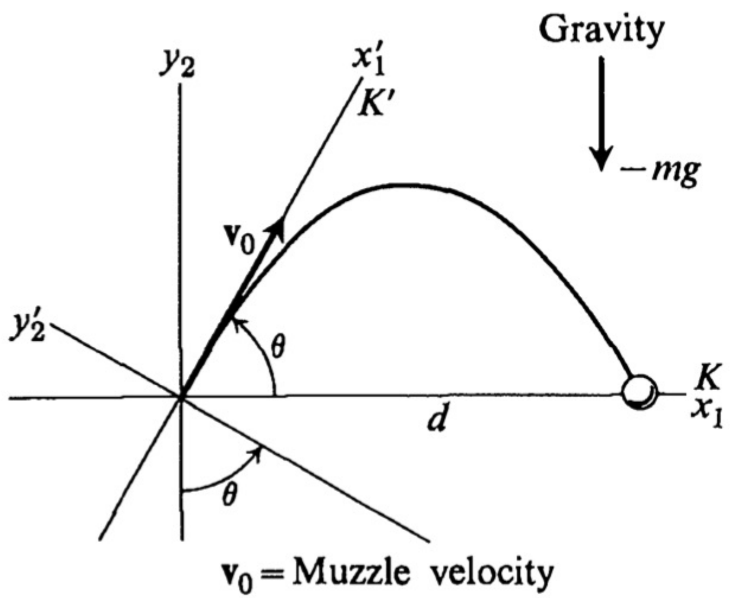
$$\boldsymbol{F} = m\boldsymbol{a} \rightarrow F_i = ma_i \rightarrow F_i = m\dot{v}_i$$

$$\dot{\boldsymbol{a}} = \frac{d\boldsymbol{a}}{dt}$$

$$F_i = m\dot{v}_i = m\ddot{x}_i$$

$$v_i = \frac{dx_i}{dt} = \dot{x}_i$$

$$a_i = \dot{v}_i = \frac{dv_i}{dt} = \frac{d\dot{x}_i}{dt} = \ddot{x}_i$$



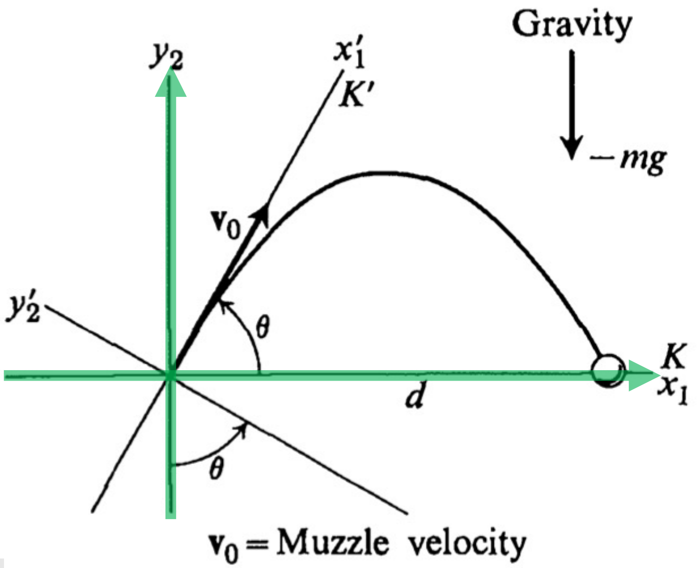
Let's assume acceleration $\ddot{x}_i$ is function of time, so that

$$\ddot{\boldsymbol{x}} \equiv \ddot{x}_i(t)$$

Furthermore, if we assume the mass is constant (which is quite usual), the second law is equation with the location $\boldsymbol{x}_i$ and its derivatives as variable – do not forget another variable time (t).

Newton's second law

$$F_i(t) = m\,\ddot{x}_i(t)$$



Let's use K coordinate system

1. Initial condition in terms of location (x_i) and velocity ($\dot{x}_i$):

$$x_i(0) = 0, \qquad \text{with } i = 1,2$$
$$\dot{x}_1(0) = v_0 \cos\theta$$
$$\dot{x}_2(0) = v_0 \sin\theta$$

$x_i(0)$ means $x_i(t = 0)$

2. Force given by gravity is constant (gravity field):

$$F_1 = m\ddot{x}_1 = 0, \qquad F_2 = -mg = m\ddot{x}_2$$

3. Estimate $x_i(\mathrm{t})$ =?

$$x_i(\mathrm{t}) = x_i(0) + \int_0^t \frac{dx_i}{dt}dt = \mathrm{x_i}(0) + \int_0^t \dot{x}_i dt$$

$$\dot{x}_i(t) = \dot{x}_i(0) + \int_0^t \frac{d\dot{x}_i}{dt}dt$$

$$\dot{x}_1(t) = \dot{x}_1(t=0) + \int_0^t \ddot{x}_1 dt = v_0\cos\theta + 0$$

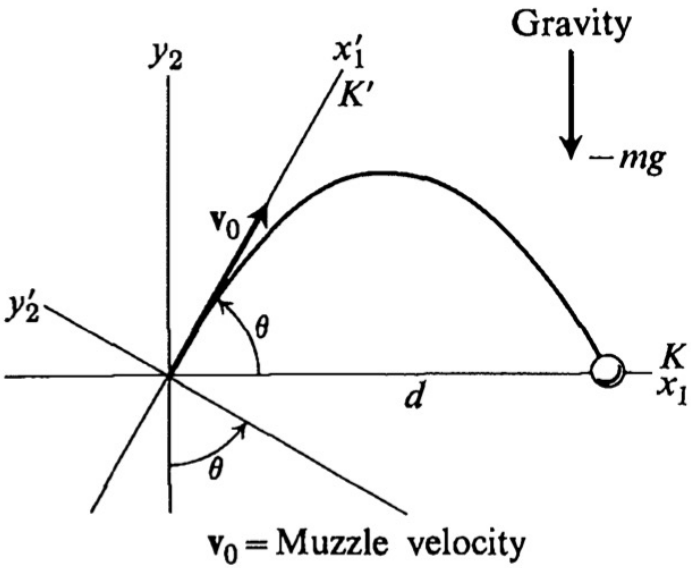
$$\dot{x}_2(t) = \dot{x}_2(t=0) + \int_0^t \ddot{x}_2 dt = v_0\sin\theta + \int_0^t -g dt = v_0\sin\theta - gt$$

$$x_1(t) = \int_0^t v_0\cos\theta\, dt = v_0 t\cos\theta$$

$$x_2(t) = \int_0^t (v_0\sin\theta - gt)dt = v_0 t\sin\theta - \frac{1}{2}gt^2$$

Newton's second law

$$F_i(t) = m\ddot{x}_i(t)$$



Let's use K' coordinate system

1. Initial condition in terms of location (x_i) and velocity ($\dot{x}_i$):

$$x'_i(t=0) = 0, \qquad \text{with } i = 1,2$$

$$\dot{x}'_1(0) = v_0$$

$$\dot{x}'_2(0) = 0$$

2. Force given by gravity is constant (gravity field):

$$F_1 = m\ddot{x}_1 = -mg\sin\theta\,, \qquad F_2 = -mg\cos\theta = m\ddot{x}_2$$

3. Estimate $x_i(\mathrm{t}) =$?

$$x_i(\mathrm{t}) = x_i(0) + \int_0^t \frac{dx_i}{dt}dt = \mathrm{x_i}(0) + \int_0^t \dot{x}_i dt$$

$$\dot{x}_i(t) = \dot{x}_i(0) + \int_0^t \ddot{x}_i dt$$

$$\dot{x}_1(t) = \dot{x}_1(0) + \int_0^t \ddot{x}_1 dt = v_0 - \int_0^t g\sin\theta\, dt = v_0 - gt\sin\theta$$

$$\dot{x}_2(t) = \dot{x}_2(0) + \int_0^t \ddot{x}_2 dt = 0 - \int_0^t g\cos\theta\, dt = -gt\cos\theta$$

$$x_1(t) = \int_0^t (v_0 - gt\sin\theta)dt = v_0 t - \frac{1}{2}gt^2\sin\theta$$

$$x_2(t) = \int_0^t -gt\cos\theta\, dt = -\frac{1}{2}gt^2\cos\theta$$

Graphing the two results.

Plot the result with theta=45 degree

At t=0

at t=1s

at t=10s

at t=100s

Plot the result with theta=90 degree

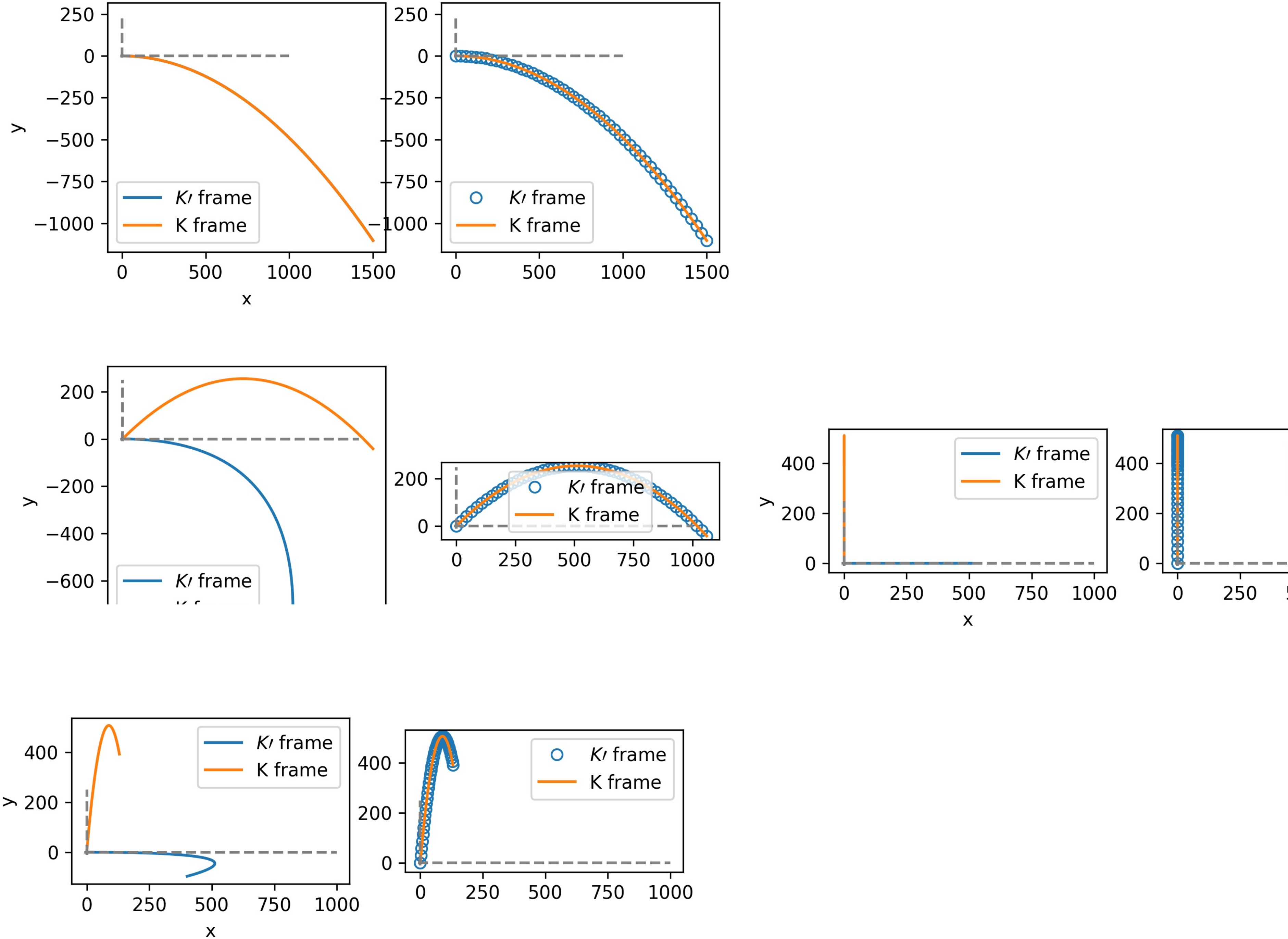
At t=0

at t=1s

at t=10s

at t=100s

- Which of the frame was the easy one?
- Describe why we'd want to chose a frame that gives easy calculation?


K′ frame
K frame
x
y

Summary

- Nomenclature
- What vectorial quantity is required?
- Vector operations (addition, scalar multiplication, inner dot)
  - Use the same coordinate system for vector operations
- Dyadic operation and Schmid tensor
- Identity matrix (Kronecker delta)
- Transpose operation
- Matrix addition and multiplication
- Changes of basis

Reference

https://www.continuummechanics.org