Finite strain

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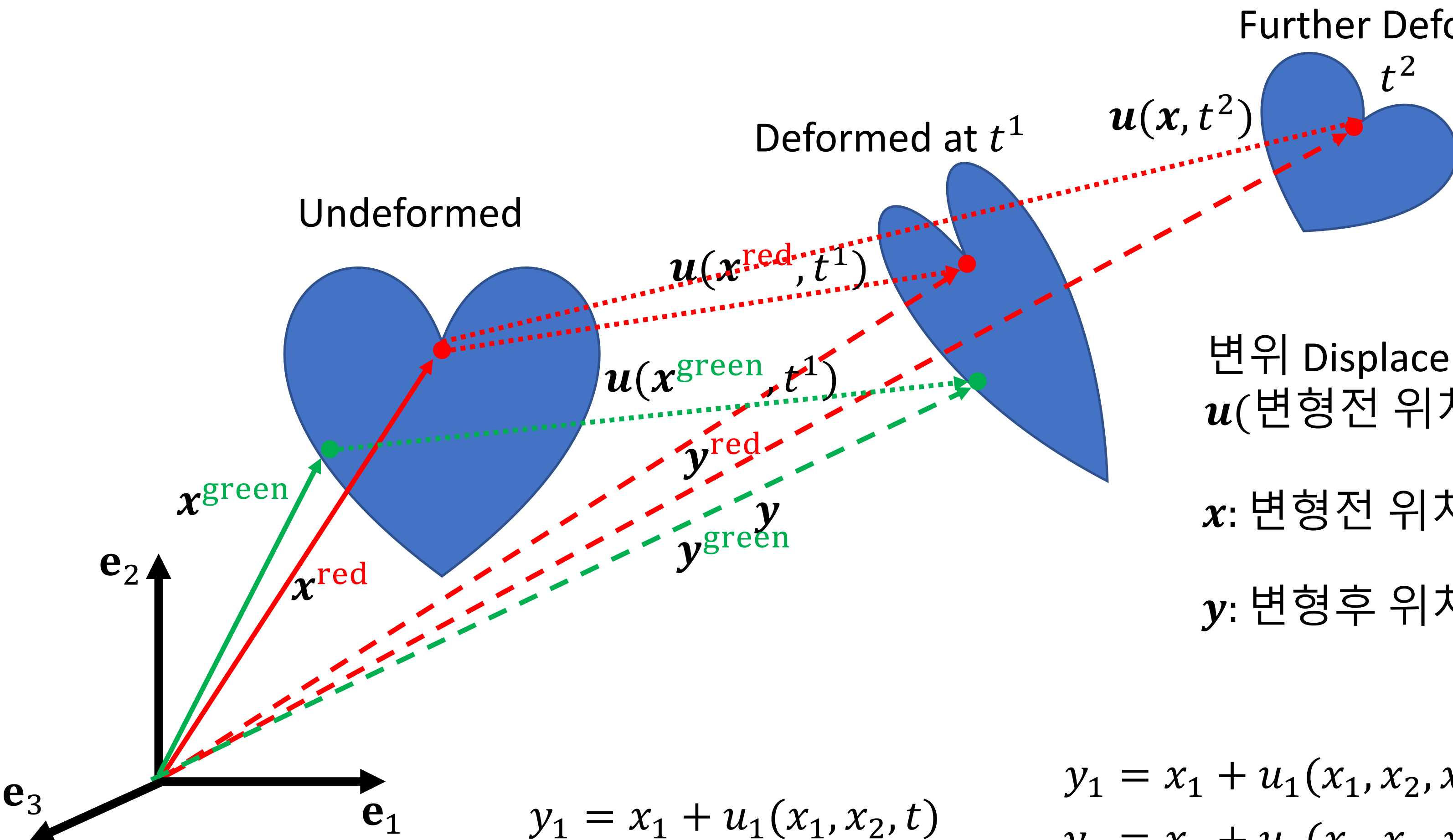
Field variables

- Scalar field
  - temperature field, density field, etc.
- Vector field
  - Force field …
- Tensor field
  - strain field, stress field …

Displacement fields



변위 Displacement:
$\boldsymbol{u}$(변형전 위치, 시간)

$\boldsymbol{x}$: 변형전 위치 벡터

$\boldsymbol{y}$: 변형후 위치 벡터

$$y_1 = x_1 + u_1(x_1, x_2, x_3, t)$$
$$y_2 = x_2 + u_2(x_1, x_2, x_3, t)$$
$$y_3 = x_3 + u_3(x_1, x_2, x_3, t)$$

$$y_1 = x_1 + u_1(x_1, x_2, t)$$
$$y_2 = x_2 + u_2(x_1, x_2, t)$$

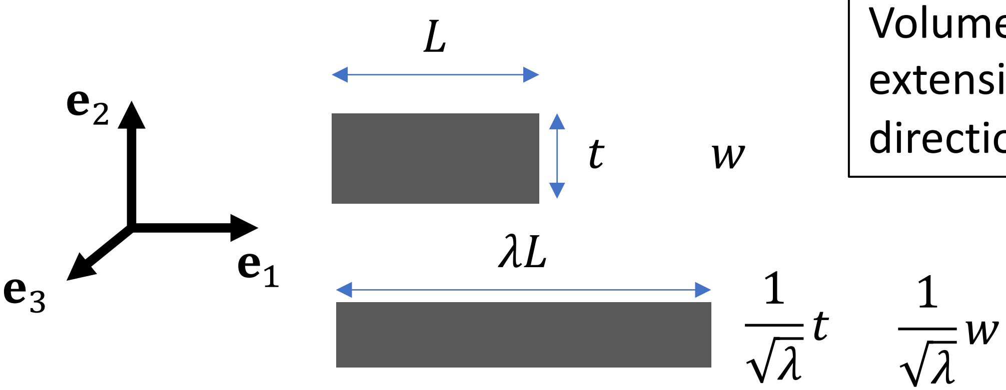
$$\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x}, t)$$

$$\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{u}(x_1, x_2, t)$$

$$\boxed{y_i = x_i + u_i(x_1, x_2, x_3, t)}$$

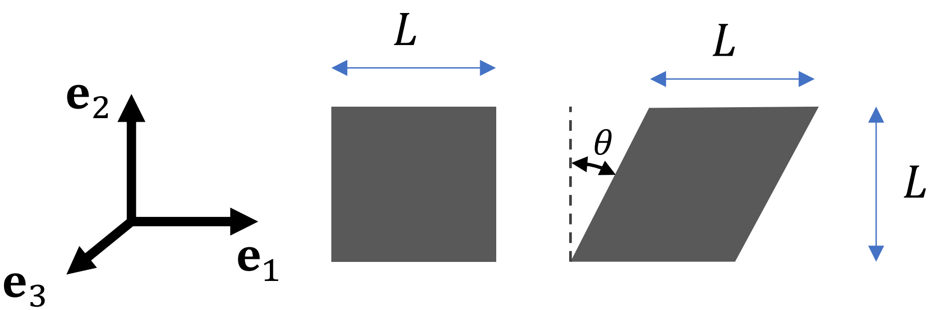
Examples of homogeneous **motion**

$$y_1 = x_1 + u_1(x_1, x_2, x_3, t)$$
$$y_2 = x_2 + u_2(x_1, x_2, x_3, t)$$
$$y_3 = x_3 + u_3(x_1, x_2, x_3, t)$$



Volume preserving uniaxial extension along the direction parallel with $\mathbf{e}_1$

$$y_1 = \lambda x_1$$
$$y_2 = x_2/\sqrt{\lambda}$$
$$y_3 = x_3/\sqrt{\lambda}$$



Simple shear

$$y_1 = x_1 + \tan\theta\, x_2$$
$$y_2 = x_2$$
$$y_3 = x_3$$

General homogeneous **motion**

$$y_1 = ax_1 + bx_2 + cx_3 + d$$
$$y_2 = ex_1 + fx_2 + gx_3 + h$$
$$y_3 = ix_1 + jx_2 + kx_3 + l$$

$$y_1 = A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + c_1$$
$$y_2 = A_{21}x_1 + A_{22}x_2 + A_{23}x_3 + c_2$$
$$y_3 = A_{31}x_1 + A_{32}x_2 + A_{33}x_3 + c_3$$

$$y_i = \sum_j A_{ij}x_j + c_i \qquad y_i = A_{ij}x_j + c_i$$

$$\boldsymbol{y} = \boldsymbol{A} \cdot \boldsymbol{x} + \boldsymbol{c}$$

$$[\boldsymbol{y}] = [\boldsymbol{A}][\boldsymbol{x}] + [\boldsymbol{c}]$$

‘Linear’ transform

General homogeneous **motion**

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‘Linear’ transform

Displacement Gradient and Deformation Gradient tensors

$$F_{ij} = \delta_{ij} + \frac{\partial u_j}{\partial x_j}$$

δ_{ij} 는 Kronecker delta 라 불리며 다음의 성질을 따른다.

If $i = j, \delta_{ij} = 1$

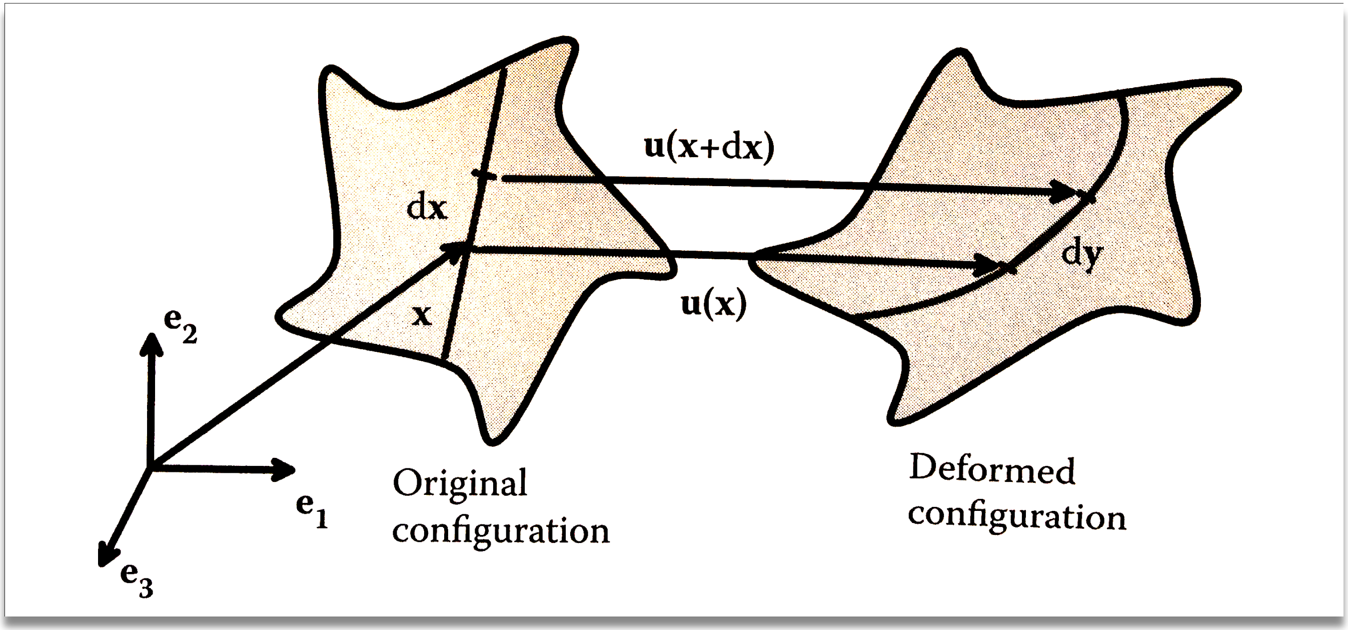
If $i \neq j, \delta_{ij} = 0$

$$\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x}, t)$$

$$\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}} = \frac{\partial(\boldsymbol{x} + \boldsymbol{u})}{\partial \boldsymbol{x}}$$

$$\frac{\partial y_i}{\partial x_j} = \frac{\partial(x_i + u_i)}{\partial x_j} = \delta x_{ij} + \frac{\partial u_i}{\partial x_j} = F_{ij}$$

Inhomogeneous motion

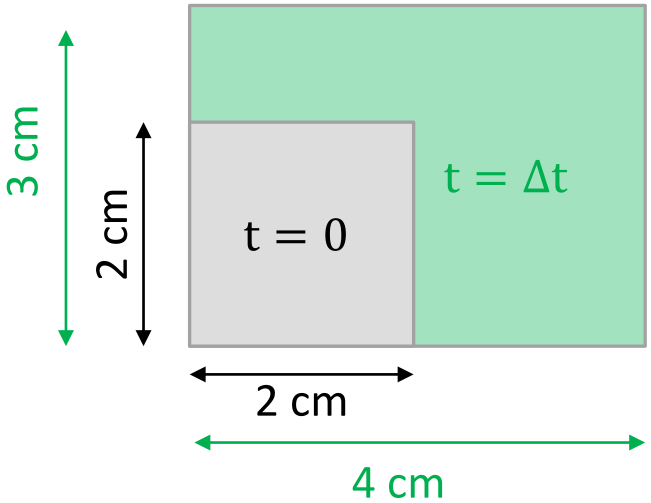


$$dy = F \cdot dx \quad \text{or} \quad dy_i = F_{ij} dx_j$$

$$\begin{bmatrix} dy_1 \\ dy_2 \\ dy_3 \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} \qquad \begin{bmatrix} dy_1 \\ dy_2 \\ dy_3 \end{bmatrix} = \begin{bmatrix} 1 + \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & 1 + \frac{\partial u_2}{\partial x_2} & 1 + \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & 1 + \frac{\partial u_3}{\partial x_3} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

예제

한 물체가 다음과 같이 어떠한 운동에 의해 ‘**균일하게**’ 변형이 되었다.



Displacement gradient tensor

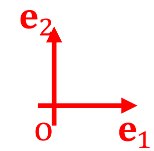
$$d_{11} = \frac{\partial u_1}{\partial x_1} = \frac{4-2}{2}$$

$$d_{22} = \frac{\partial u_2}{\partial x_2} = \frac{3-2}{2}$$

$$d_{21} = \frac{\partial u_2}{\partial x_1} = 0$$

$$d_{12} = \frac{\partial u_1}{\partial x_2} = 0$$

한 점의 좌표: (x_1, x_2)



2차원 좌표계
($\mathbf{e}_1, \mathbf{e}_2$ basis vectors)

$$d_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$F_{ij} = d_{ij} + \delta_{ij}$$

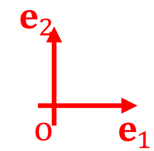
$$= \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 1.5 \end{bmatrix}$$

예제

한 물체가 다음과 같이 어떠한 운동에 의해 '**균일하게**' 변형이 되었다.

한 점의 좌표: (x_1, x_2)



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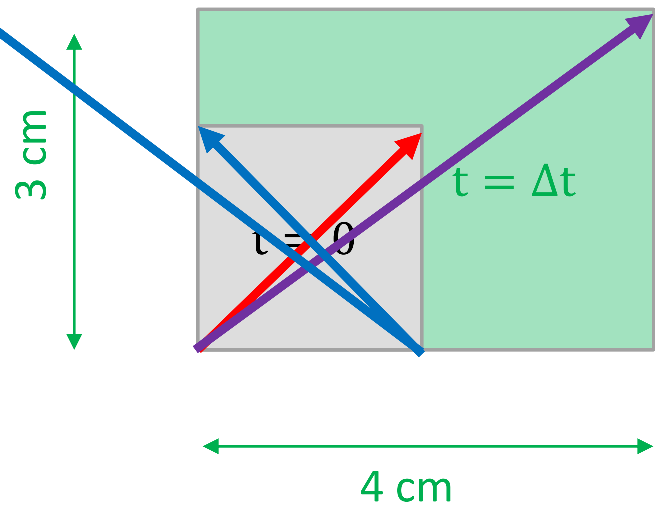
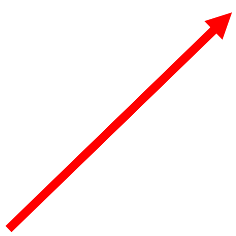


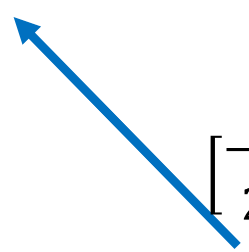
$$d_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$F_{ij} = \begin{bmatrix} 2 & 0 \\ 0 & 1.5 \end{bmatrix}$$



$$\boldsymbol{l} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$F_{ij} l_j = \begin{bmatrix} 2 & 0 \\ 0 & 1.5 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \end{bmatrix}$$



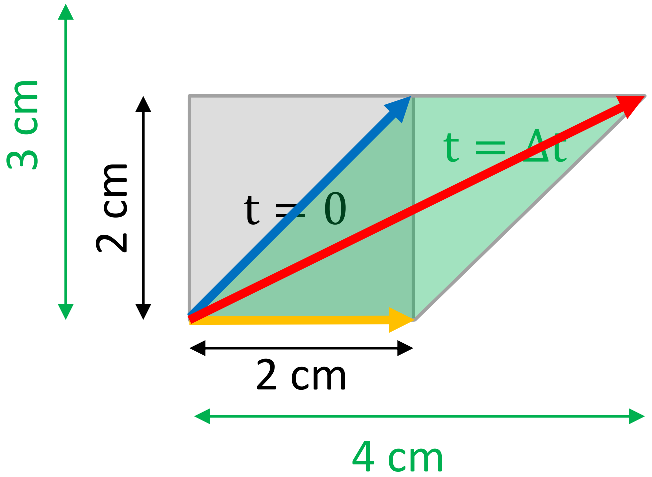
$$\begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$F_{ij} l_j = \begin{bmatrix} 2 & 0 \\ 0 & 1.5 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 & 3 \end{bmatrix}$$

F_{ij} does not account for 'translation'

예제

$$F_{ij} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}$$



$$\xrightarrow{[2,0]} \quad \mathrm{F_{ij}} \quad \xrightarrow{[2,0]}$$

$$\begin{bmatrix} \mathrm{F_{11}} & \mathrm{F_{12}} \\ \mathrm{F_{21}} & \mathrm{F_{22}} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \rightarrow \quad 2 = 2\mathrm{F_{11}}, 0 = 2\mathrm{F_{21}}$$

$$\rightarrow \mathrm{F_{11}} = 1, \mathrm{F_{21}} = 0$$

$$\mathrm{F_{ij}} = \begin{bmatrix} 1 & \mathrm{F_{12}} \\ 0 & \mathrm{F_{22}} \end{bmatrix}$$

$$\xrightarrow{[2,2]} \quad \mathrm{F_{ij}} \quad \xrightarrow{[4,2]}$$

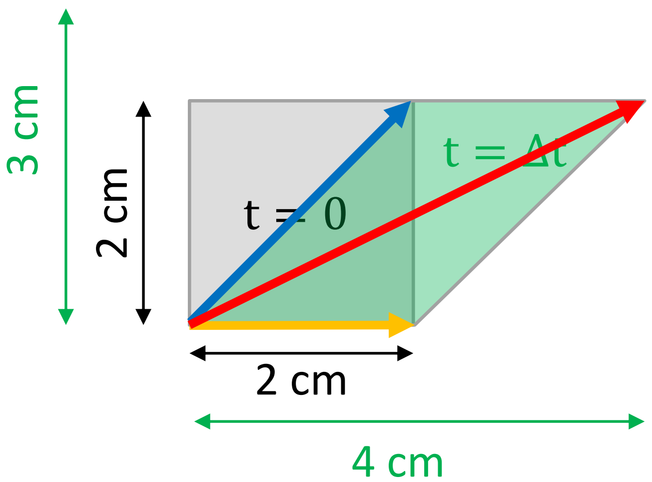
$$\begin{bmatrix} 1 & \mathrm{F_{12}} \\ 0 & \mathrm{F_{22}} \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \qquad \begin{aligned} 2 + 2\mathrm{F_{12}} &= 4 \\ 2\mathrm{F_{22}} &= 2 \end{aligned}$$

$$\mathrm{F_{ij}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Q. Can you find $\mathrm{d_{ij}}$ for this deformation?

예제

$$F_{ij} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}$$



$$\xrightarrow{[2,0]} \quad F_{ij} \quad \xrightarrow{[2,0]}$$

$$\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \rightarrow \quad 2 = 2F_{11}, 0 = 2F_{21}$$

$$\rightarrow F_{11} = 1, F_{21} = 0$$

$$F_{ij} = \begin{bmatrix} 1 & F_{12} \\ 0 & F_{22} \end{bmatrix}$$

$$\xrightarrow{[2,2]} \quad F_{ij} \quad \xrightarrow{[4,2]}$$

$$\begin{bmatrix} 1 & F_{12} \\ 0 & F_{22} \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

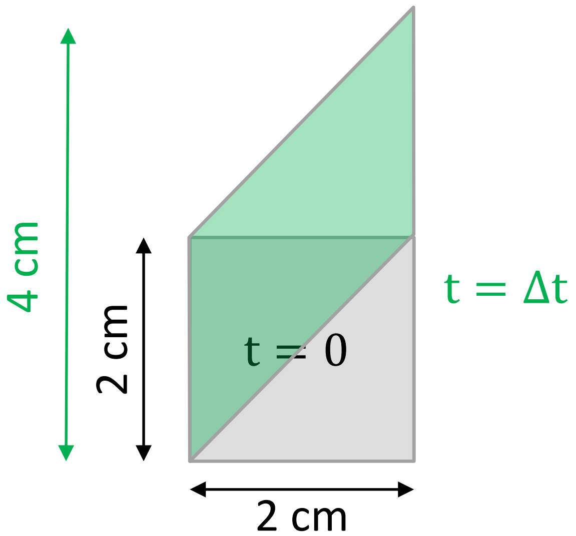
$$2 + 2F_{12} = 4$$

$$2F_{22} = 2$$

$$F_{ij} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

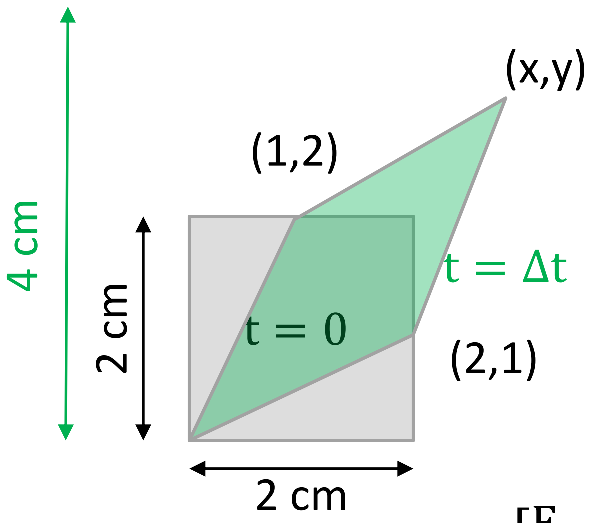
Q. Can you find d_{ij} for this deformation?

예제



Find out the deformation gradient for the given deformation illustrated in the left figure.

예제



Find out the deformation gradient for the given deformation illustrated in the left figure.

$$\begin{bmatrix} \mathrm{F}_{11} & \mathrm{F}_{12} \\ \mathrm{F}_{21} & \mathrm{F}_{22} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightarrow \begin{matrix} 2 = 2F_{11} \\ 1 = 2F_{21} \end{matrix} \qquad \begin{matrix} F_{11} = 1 \\ F_{21} = 0.5 \end{matrix}$$

$$\begin{bmatrix} \mathrm{F}_{11} & \mathrm{F}_{12} \\ \mathrm{F}_{21} & \mathrm{F}_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightarrow \begin{matrix} 2 = 2F_{22} \\ 1 = 2F_{12} \end{matrix} \qquad \begin{matrix} F_{22} = 1 \\ F_{12} = 0.5 \end{matrix}$$

$$\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Polar decomposition

- Deformation gradient can be decomposed into
  - Stretch
  - Rotation