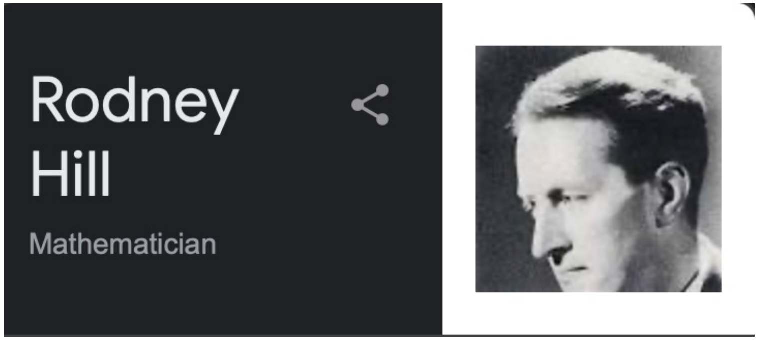
Plasticity (described with tensors)

Youngung Jeong

Theory of plasticity

- Rodney Hill



Yield criterion with stress tensor (not scalar)

- In order to use tensorial quantities and apply the former method, we'll need to adjust a few assumptions.
- The use of Heaviside like function for yield criterion

$$\widehat{H}(\sigma - Y) \rightarrow \widehat{H}(\boldsymbol{\sigma}, Y). \qquad \widehat{H}\left(\sigma_{ij}, Y\right).$$

We cannot just subtract $\sigma_{ij} - Y$: σ_{ij} is a tensor, Y is a scalar quantity.

We introduce a scalar function, called the yield function.
Yield criterion is described as a function of σ_{ij} (two free indices)

$\phi = \phi\left(\sigma_{ij}\right)$ and Let's use ϕ to see if yield condition is met ($\phi = Y$) or not ($\phi < Y$).

Strain-hardening with stress, strain tensors (not scalars)

- One use length of vector to quantify the ‘size’ of a vector.
- Similarly, we use equivalent scalar quantities for stress and strain tensors.

- The equivalent scalar quantity for stress tensor is simply called ‘equivalent stress’, and the same is applied to strain tensor (’equivalent strain’).

- There are a few types of equivalent quantities. We’ll use only von Mises quantity.

$$\bar{\sigma}^{eq} = \left[\frac{1}{2}\{(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2)\}\right]^{0.5}$$

- The definition of equivalent strain is not straightforward. We’ll have to determine the amount of plastic work done to the material that is under the given stress σ_{ij} (thus providing us the above equivalent stress tensor).

Plastic work done

- Apparently, plasticity is non-linear, the work done for the time from 0 to t is defined using integration:
- $w^{pl}(t) = w^{pl}(t=0) + \int_0^t dw$
- $dw^{pl} = \boldsymbol{\sigma}: d\boldsymbol{\varepsilon}^{pl} = \sigma_{ij} d\varepsilon_{ij}^{pl}$ (two dummy indices i, j)
- We postulate the work done calculated by stress and strain tensor (as done above) should be the same as the one calculated by the equivalent stress, equivalent strain.
- The above postulation is expressed as below:
- $dw^{pl} = \sigma_{ij} d\varepsilon_{ij}^{pl} = \sigma^{eq} d\varepsilon^{eq} \rightarrow d\varepsilon^{eq} = \frac{\sigma_{ij} d\varepsilon_{ij}^{pl}}{d\sigma^{eq}}$

$$\varepsilon^{eq}(t) = \varepsilon^{eq}(t=0) + \int_0^t d\varepsilon^{eq} = \varepsilon^{eq}(t=0) + \int_0^t \frac{\sigma_{ij} d\varepsilon_{ij}^{pl}}{d\sigma^{eq}}$$

- For materials without previous deformation, the above can be written as:

$$\varepsilon^{eq}(t) = \int_0^t \frac{\boldsymbol{\sigma}: d\boldsymbol{\varepsilon}^{pl}}{d\sigma^{eq}}$$

Strain hardening?

- Say, your material obeys the Hollomon equation,

$$\sigma = K'\varepsilon^n$$

$\boldsymbol{\sigma} = K'\boldsymbol{\varepsilon}^n$ (not possibly); The mathematical operation of 'exponential' is not available.

$\sigma^{eq} = K'(\varepsilon^{eq})^n$; Instead of using the tensors, we use 'equivalent' quantities (that are scalars).

- In case your material obeys linear hardening:

$$\sigma^{eq} = K\varepsilon^{eq}$$

Now, let's look at the constitutive model again

$$\frac{d\boldsymbol{\varepsilon}}{dt} = c\hat{H}\left\{\boldsymbol{\sigma} - k\left(\boldsymbol{\varepsilon}(t) - \int_0^t \frac{1}{\mathbb{E}} : \frac{d\boldsymbol{\sigma}}{dt} dt\right)\right\} + \frac{1}{\mathbb{E}} : \frac{d\boldsymbol{\sigma}}{dt}$$

The term for strain hardening $k\left(\boldsymbol{\varepsilon}(t) - \int_0^t \frac{1}{\mathbb{E}} \frac{d\mathbb{E}}{dt} dt\right)$ should be replaced by the empirical laws based on equivalent quantities.

$$\boldsymbol{\sigma} - k\left(\boldsymbol{\varepsilon}(t) - \int_0^t \frac{1}{\mathbb{E}} : \frac{d\boldsymbol{\sigma}}{dt} dt\right) \rightarrow \phi(\boldsymbol{\sigma}) - \sigma^{eq}(\varepsilon^{eq}) \qquad \varepsilon^{eq}(t) = \int_0^t \frac{\boldsymbol{\sigma} : d\boldsymbol{\varepsilon}}{d\sigma^{eq}}$$

c should be somehow replaced with a tensorial quantity similar to $\frac{d\boldsymbol{\varepsilon}^{pl}}{dt}$

Now, let's look at the constitutive model again

$$\frac{d\boldsymbol{\varepsilon}}{dt} = c\widehat{H}\left\{\boldsymbol{\sigma} - k\left(\boldsymbol{\varepsilon}(t) - \int_0^t \frac{1}{\mathbb{E}}:\frac{d\boldsymbol{\sigma}}{dt}dt\right)\right\} + \frac{1}{\mathbb{E}}:\frac{d\boldsymbol{\sigma}}{dt} \qquad \frac{d\boldsymbol{\varepsilon}}{dt} = \frac{d\boldsymbol{\varepsilon}^{pl}}{dt}\widehat{H}(\phi(\boldsymbol{\sigma}) - K'\varepsilon^{eq}) + \frac{1}{\mathbb{E}}:\frac{d\boldsymbol{\sigma}}{dt}$$

Let's assume the linear strain hardening ($\sigma^{eq} = K'\varepsilon^{eq}$) is valid for our material, so that $k\left(\boldsymbol{\varepsilon}(t) - \int_0^t \frac{1}{\mathbb{E}}:\frac{d\boldsymbol{\sigma}}{dt}dt\right)$ should be replaced by $K'\varepsilon^{eq}$

$$\boldsymbol{\sigma} - k\left(\boldsymbol{\varepsilon}(t) - \int_0^t \frac{1}{\mathbb{E}}:\frac{d\boldsymbol{\sigma}}{dt}dt\right) \rightarrow \phi(\boldsymbol{\sigma}) - K'\varepsilon^{eq} \qquad \varepsilon^{eq}(t) = \int_0^t \frac{\boldsymbol{\sigma}: d\boldsymbol{\varepsilon}}{d\sigma^{eq}}$$

c should be somehow replaced with a tensorial quantity similar to $\frac{d\boldsymbol{\varepsilon}^{pl}}{dt}$, for now, Let's replace c with $\frac{d\boldsymbol{\varepsilon}^{pl}}{dt}$ as $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{pl} + \boldsymbol{\varepsilon}^{el}$. $\left(\rightarrow \frac{d\boldsymbol{\varepsilon}}{dt} = \frac{d\boldsymbol{\varepsilon}^{pl}}{dt} + \frac{d\boldsymbol{\varepsilon}^{el}}{dt}\right)$

Finding unknown plastic strain $\frac{d\varepsilon^{eq}}{dt}$

$$\frac{d\boldsymbol{\varepsilon}}{dt} = \frac{d\boldsymbol{\varepsilon}^{pl}}{dt}\widehat{H}(\phi(\boldsymbol{\sigma}) - K'\varepsilon^{eq}) + \frac{1}{\mathbb{E}} : \frac{d\boldsymbol{\sigma}}{dt}$$

$$\frac{d\boldsymbol{\varepsilon}}{dt} = \frac{d\boldsymbol{\varepsilon}^{pl}}{dt}\widehat{H}(\phi(\boldsymbol{\sigma}) - K'\varepsilon^{eq}) + \frac{1}{\mathbb{E}} : \frac{d\boldsymbol{\sigma}}{dt} \qquad \frac{d\boldsymbol{\varepsilon}}{dt} = \frac{d\boldsymbol{\varepsilon}^{pl}}{dt} + \frac{1}{\mathbb{E}} : \frac{d\boldsymbol{\sigma}}{dt}$$

$$\frac{d\boldsymbol{\varepsilon}^{pl}}{dt} ??$$

Let's postulate Hill's idea (associated flow rule)

$$\frac{d\boldsymbol{\varepsilon}^{pl}}{dt} = \frac{d\varepsilon^{eq}}{dt}\left(\frac{\partial\phi(\boldsymbol{\sigma})}{\partial\boldsymbol{\sigma}}\right)$$

If we use von Mises yield criterion:

$$\frac{d\varepsilon_{ij}^{pl}}{dt} = \frac{d\varepsilon^{eq}}{dt}\left(\frac{\partial\left\{\left[\frac{1}{2}\{(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2)\}\right]^{0.5}\right\}}{\partial\sigma_{ij}}\right)$$

If we use von Mises yield criterion:

$$\frac{d\varepsilon_{ij}^{pl}}{dt} = \frac{d\left\{\int_0^t \frac{\boldsymbol{\sigma}:d\boldsymbol{\varepsilon}}{d\sigma^{eq}}\right\}}{dt}\left(\frac{\partial\left\{\left[\frac{1}{2}\{(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2)\}\right]^{0.5}\right\}}{\partial\sigma_{ij}}\right)$$

If we use von Mises yield criterion:

$$\frac{d\varepsilon_{ij}^{pl}}{dt} = \frac{d\left\{\int_0^t \frac{\boldsymbol{\sigma}:d\boldsymbol{\varepsilon}}{d\sigma^{eq}}\right\}}{dt}\left(\frac{\partial\left\{\left[\frac{1}{2}\{(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2+6(\sigma_{12}^2+\sigma_{23}^2+\sigma_{13}^2)\}\right]^{0.5}\right\}}{\partial\sigma_{ij}}\right)$$

$$\frac{\partial\left\{\left[\frac{1}{2}\{(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2+6(\sigma_{12}^2+\sigma_{23}^2+\sigma_{13}^2)\}\right]^{0.5}\right\}}{\partial\sigma_{ij}}$$

Notice the two free indices (i,j)

Let's say,

$$X = (\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2+6(\sigma_{12}^2+\sigma_{23}^2+\sigma_{13}^2)$$

Then,

$$\frac{\partial\left\{\left[\frac{1}{2}X\right]^{0.5}\right\}}{\partial\sigma_{ij}} = \frac{\partial\left\{\left[\frac{1}{2}X\right]^{0.5}\right\}}{\partial X}\frac{\partial X}{\partial\sigma_{ij}} = \frac{1}{2}^{0.5}\frac{\partial\{[X]^{0.5}\}}{\partial X}\frac{\partial X}{\partial\sigma_{ij}} = \frac{1}{2}^{0.5} 0.5\, X^{-0.5}\frac{\partial X}{\partial\sigma_{ij}} = 0.5\left(\frac{1}{2X}\right)^{0.5}\frac{\partial X}{\partial\sigma_{ij}}$$

$$\frac{\partial X}{\partial\sigma_{11}} = 2(\sigma_{11}-\sigma_{22})+2(\sigma_{11}-\sigma_{33})$$

$$\frac{\partial X}{\partial\sigma_{22}} = 2(\sigma_{22}-\sigma_{11})+2(\sigma_{22}-\sigma_{33})$$

$$\frac{\partial X}{\partial\sigma_{33}} = 2(\sigma_{33}-\sigma_{11})+2(\sigma_{33}-\sigma_{22})$$

$$\frac{\partial X}{\partial\sigma_{12}} = 12\sigma_{12} = \frac{\partial X}{\partial\sigma_{21}}$$

$$\frac{\partial X}{\partial\sigma_{23}} = 12\sigma_{23} = \frac{\partial X}{\partial\sigma_{32}}$$

$$\frac{\partial X}{\partial\sigma_{13}} = 12\sigma_{13} = \frac{\partial X}{\partial\sigma_{31}}$$

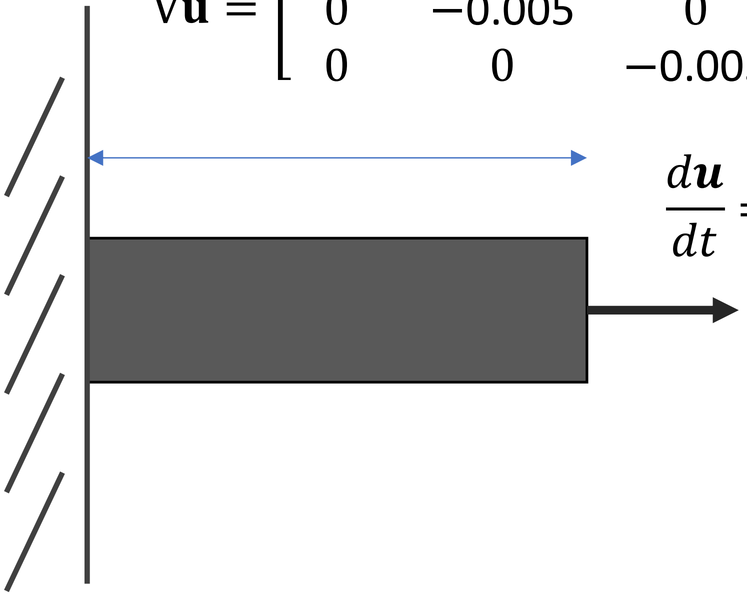
Problem: stretching an elasto-plastic rod with linear hardening

$$\frac{d\boldsymbol{\varepsilon}}{dt} = \frac{d\boldsymbol{\varepsilon}^{pl}}{dt}\widehat{H}(\phi(\boldsymbol{\sigma}) - (Y^0 + k\varepsilon^{eq})) + \frac{1}{\mathbb{E}}:\frac{d\boldsymbol{\sigma}}{dt}$$

Perfect-plastic (no hardening; Y^0 is constant)

with associated flow rule: $\frac{d\varepsilon^{eq}}{dt}\left(\frac{\partial\phi(\boldsymbol{\sigma})}{\partial\boldsymbol{\sigma}}\right)$

$$\nabla\mathbf{u} = \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & -0.005 & 0 \\ 0 & 0 & -0.005 \end{bmatrix}$$



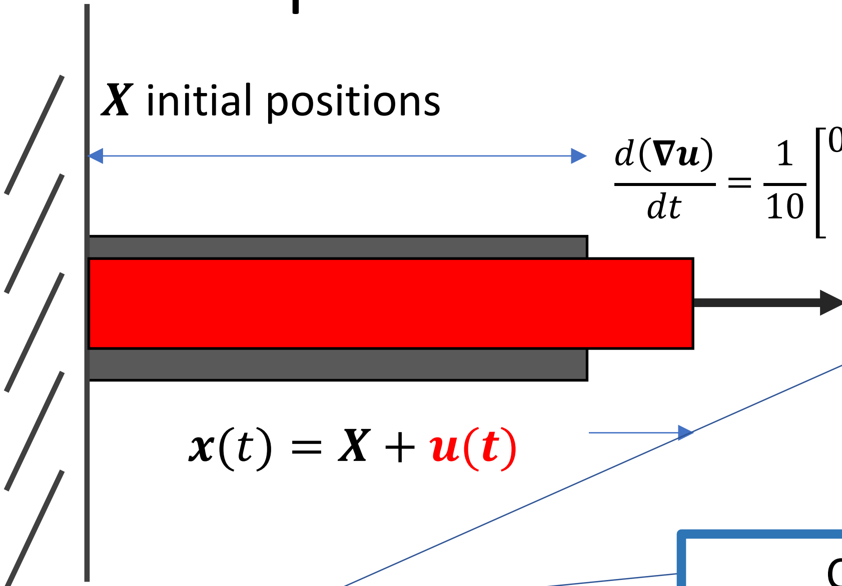
$$\frac{d\boldsymbol{u}}{dt} = \frac{\nabla\boldsymbol{u}}{10\ [sec]}$$

Let's assum your material obeys the von Mises yield criterion

$$\boldsymbol{\varepsilon} = \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & -0.005 & 0 \\ 0 & 0 & -0.005 \end{bmatrix}$$

$$\frac{d\boldsymbol{\varepsilon}}{dt} = \begin{bmatrix} 0.001 & 0 & 0 \\ 0 & -0.0005 & 0 \\ 0 & 0 & -0.0005 \end{bmatrix}$$

Elastic predictor and corrector algorithm



$$\frac{d(\nabla \boldsymbol{u})}{dt} = \frac{1}{10}\begin{bmatrix} 0.01 & 0 & 0 \\ 0 & -0.005 & 0 \\ 0 & 0 & -0.005 \end{bmatrix}$$

While time increment is Δt

$$\Delta \boldsymbol{\varepsilon} = \frac{\mathrm{d}\left\{\frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)\right\}}{dt}\Delta t$$

Initially assuming that all strain is elastic

$$\Delta \boldsymbol{\sigma} = \mathbb{E}:\Delta \boldsymbol{\varepsilon}$$

Stress is updated:

$$\boldsymbol{\sigma}_{(n+1)} = \boldsymbol{\sigma}_{(n)} + \Delta \boldsymbol{\sigma}$$

$$\boldsymbol{\sigma}_{(n+1)} = \boldsymbol{\sigma}_{(n)} + \mathbb{E}:\Delta \boldsymbol{\varepsilon}$$

$$= \boldsymbol{\sigma}_{(n)} + \mathbb{E}:\Delta \boldsymbol{\varepsilon}$$

Correcting elastic strain $\Delta\sigma = \mathbb{E}(\Delta\varepsilon - \Delta\varepsilon^{pl})$

Check if $\sigma_{(n+1)}$ really gives only elastic contribution

$\phi(\sigma_{(n+1)}) < c$ or $\phi\left(\sigma_{(n+1)}\right) = c$ or $\phi(\sigma_{(n+1)}) > c$

You are correct about the strain decomposition, let's move on to next time increment (end)

Plasticity update

$$\Delta\varepsilon_{ij}^{pl} = \Delta\varepsilon^{eq}\frac{\partial\phi}{\partial\sigma_{ij}}$$

$$\Delta\varepsilon_{ij} = \Delta\varepsilon_{ij}^{el} + \Delta\varepsilon_{ij}^{pl}$$

Elastic predictor and corrector algorithm

Initially assuming that all strain is elastic

$$\Delta\sigma_{ij} = \mathbb{E}_{ijkl}\Delta\varepsilon_{kl}$$

Find the stress corresponding to the current stress increment

$$\boldsymbol{\sigma}_{(n+1)} = \boldsymbol{\sigma}_{(n)} + \Delta\boldsymbol{\sigma}$$

Correcting elastic strain

$$\Delta\varepsilon^{el}_{ij} = \Delta\varepsilon_{ij} - \Delta\varepsilon^{pl}_{ij}$$

That gives new stress increment

$$\Delta\sigma_{ij} = \mathbb{E}_{\mathrm{ijkl}}(\Delta\varepsilon_{kl} - \Delta\varepsilon^{pl}_{kl})$$

(adjust strain decomposition

Check if $\boldsymbol{\sigma}_{(n+1)}$ is inside, on or over the yield criterion

$$\phi\left(\boldsymbol{\sigma}_{(n+1)}\right) < Y^0 + k(\varepsilon^{eq}_{(n)} + \Delta\varepsilon^{eq})$$

or $\phi\left(\boldsymbol{\sigma}_{(n+1)}\right) = Y^0 + k(\varepsilon^{eq}_{(n)} + \Delta\varepsilon^{eq})$

or $\phi\left(\boldsymbol{\sigma}_{(n+1)}\right) > Y^0 + k(\varepsilon^{eq}_{(n)} + \Delta\varepsilon^{eq})$

Plastic strain update

$$\Delta\varepsilon^{pl}_{ij} = \Delta\varepsilon^{eq}\frac{\partial\phi}{\partial\sigma_{ij}}$$

If $\phi\left(\boldsymbol{\sigma}_{(n+1)}\right) \leq Y^0 + k(\varepsilon^{eq}_{(n)} + \Delta\varepsilon^{eq})$

$\boldsymbol{\sigma}_{(n+1)}$ is consistent with our theory.

Let's move on to next time increment