Plasticity (described with scalars)

Youngung Jeong

Linear isotropic elasticity

Elastic constitutive law (Hooke's law):

$$\mathbb{E}_{ijkl}\varepsilon_{kl} = \sigma_{ij} \qquad (linear\ elasticity)$$

$$\mathbb{E}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \qquad (isotropic\ elasticity;\ two\ constants\ \lambda, \mu)$$

Replacing $\mathbb{E}_{ijkl}$ to the Hooke's law

$$\begin{aligned}\sigma_{ij} &= \mathbb{E}_{ijkl}\varepsilon_{kl} = \lambda\delta_{ij}\delta_{kl}\varepsilon_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\varepsilon_{kl} \\ &= \lambda\delta_{ij}\varepsilon_{kk} + \mu\left(\delta_{ik}\delta_{jl}\varepsilon_{kl} + \delta_{il}\delta_{jk}\varepsilon_{kl}\right) = \lambda\delta_{ij}\varepsilon_{kk} + \mu\left(\delta_{ik}\varepsilon_{kj} + \delta_{il}\varepsilon_{jl}\right) \\ &= \lambda\delta_{ij}\varepsilon_{kk} + \mu\left(\varepsilon_{ij} + \varepsilon_{ji}\right) = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}\end{aligned}$$

History of plasticity and metal forming analysis

https://doi.org/10.1016/j.jmatprotec.2010.04.001

Journal of Materials Processing Technology 210 (2010) 1436–1454

Contents lists available at ScienceDirect

Journal of Materials Processing Technology

journal homepage: www.elsevier.com/locate/jmatprotec

History of plasticity and metal forming analysis☆

Kozo Osakada*

Osaka University, School of Engineering Science, Machikaneyama 3, Toyonaka, Osaka, Japan

A R T I C L E I N F O

Article history:
Received 6 November 2009
Received in revised form 1 April 2010
Accepted 1 April 2010

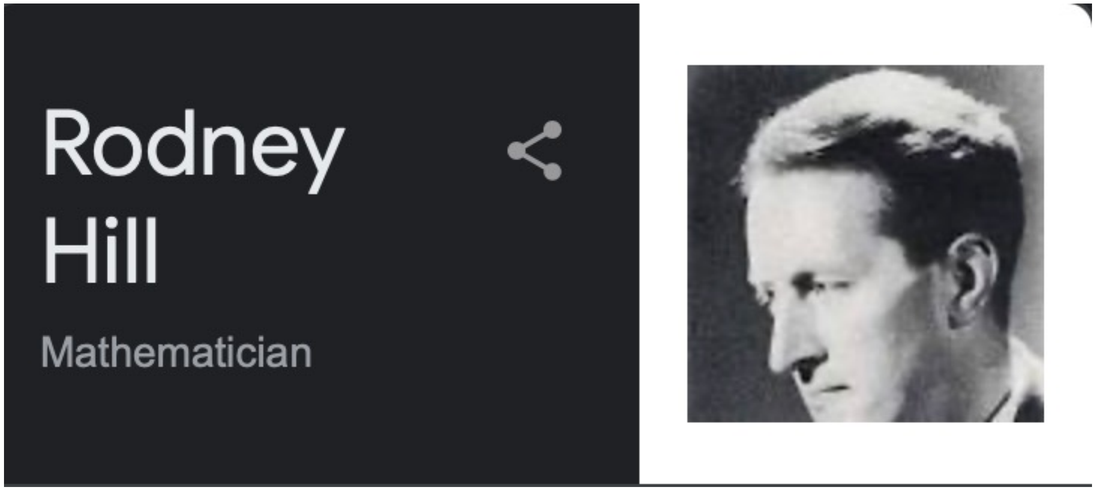
Keywords:
Metal Forming
Plasticity
History

A B S T R A C T

The research history of mechanics, physics and metallurgy of plastic deformation, and the development of metal forming analysis are reviewed. The experimental observations of plastic deformation and metal forming started in France by Coulomb and Tresca. In the early 20th century, fundamental investigation into plasticity flourished in Germany under the leadership of Prandtl, but many researchers moved out to the USA and UK when Hitler came in power. In the second half of the 20th century, some analyzing methods of metal forming processes were developed and installed onto computers as software, and they are effectively used all over the world.

Theory of plasticity

- Rodney Hill



What makes plasticity complicated

- Plasticity is more complicated than elasticity, because of the ‘non-linearity’ in the constitutive behavior.

If it were linear, as in the elasticity, the constitutive description based on tensors can be expressed as linear algebraic expression such that

$$[\boldsymbol{\sigma}] = [\mathbb{E}][\boldsymbol{\varepsilon}^{el}]$$

- Elastic deformation occurs whenever $\boldsymbol{\sigma} \neq \mathbf{0}$ is applied. In plasticity, there is a certain set of criteria which should be met for plasticity to kick in. (Yield criterion, yield surface)

- The non-linearity in the plastic constitutive description, is well expressed in its ’differential equation’.

$$\frac{d\varepsilon_{ij}}{dt} = \frac{d\lambda}{dt}\frac{\partial\phi(\boldsymbol{\sigma})}{\partial\sigma_{ij}}$$

(In the above two free indices implied: i, j; Since $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are symmetric, only 6 equations are implied.)

- Strain-hardening occurs; Strain rate-sensitivity is present.

What makes plasticity complicated

- Elastic behavior of metals (and of most materials) is very well understood. However, plasticity remains relatively not well understood and behaviors are often described *empirically* (assumptions based on observing the phenomena/experimental behaviors)

- Not a single rule prevails ... (as opposed to Hooke's law for elasticity). There are various mathematical models (theories) developed to describe plastic behavior of metals.

- Experimental methods often are limited to 'uniaxial' cases (where only a single component of stress tensor is not zero). Therefore, with using the available experimental data, it feels like using a 'scalar' clue to describe 'vectorial' (or tensorial) world…

Concept of yield criterion.

- We introduce a yield criterion as a ***scalar*** function (often denoted as ϕ) of **stress tensor** $\boldsymbol{\sigma}$

- $\phi = \phi(\boldsymbol{\sigma})$. Just like $f = f(x)$. Both functions return a 'scalar' quantity (often denoted by its own symbol such as ϕ or f). The important difference between them is that the argument is 'tensor' for the case of yield function ϕ, while the function f takes in a scalar quantity x.

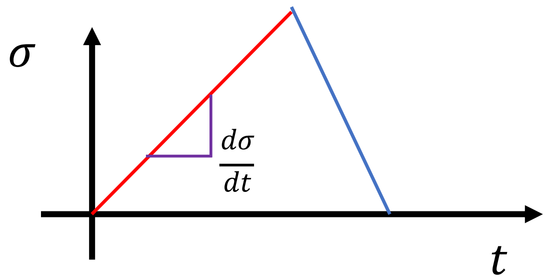
- We say if the stress state of material meets the yield condition if $\phi = c$ with c being the given material characteristic (which often represents the material's yield strength)

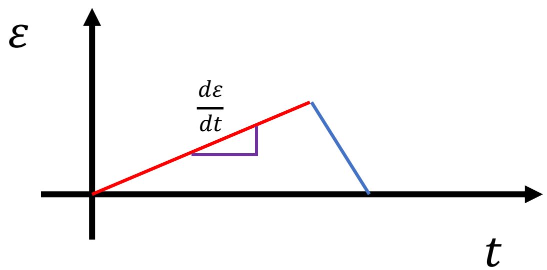
- We also use this yield function as plastic potential in flow rule (associated flow rule)

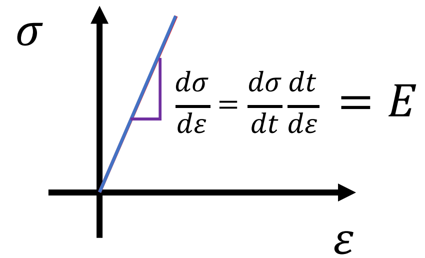
Thought experiments on 4 distinguished (virtual) materials

1. Linear elastic materials (a)
2. Rigid perfect plastic materials (b)
3. Elastic-Rigid perfect plastic material (a+b)
4. Elastic-plastic (elasto-plastic) with hardening (c)

Linear elasticity: Behavior of material that exhibits only linear elasticity (virtual experiments)







Our target (i.e., constitutive description) is to find the relationship between σ and ε

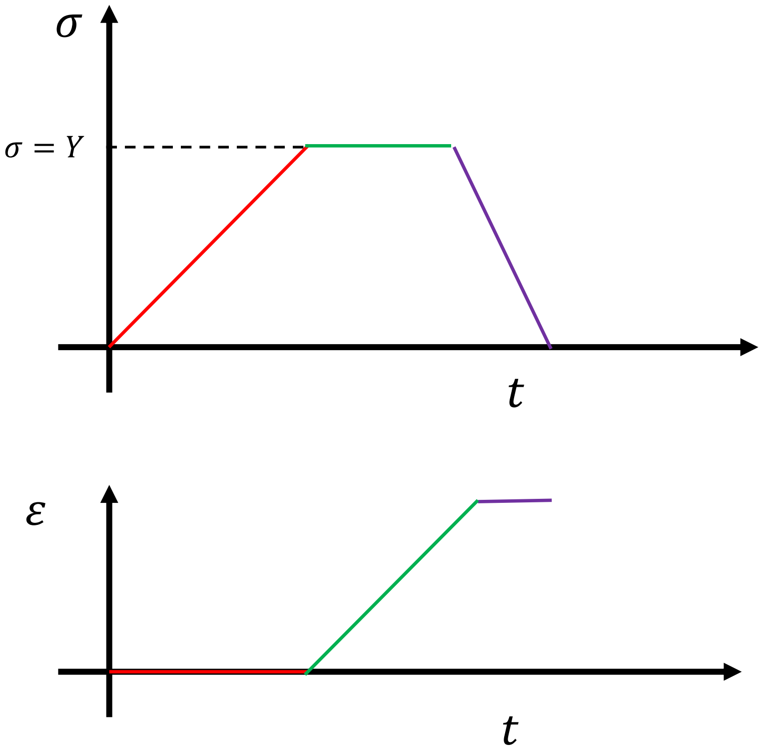
$$\frac{d\sigma}{dt} = E\frac{d\varepsilon}{dt} \rightarrow d\sigma = E d\varepsilon \rightarrow \int_0^\sigma d\sigma = E\int_0^\varepsilon d\varepsilon$$

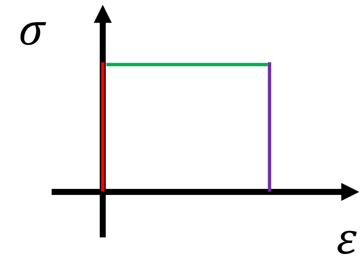
$$\rightarrow \sigma = E\varepsilon$$

With tensors, we express the above

$$\rightarrow \boldsymbol{\sigma} = \mathbb{E}\colon \boldsymbol{\varepsilon}$$

Rigid perfect plastic (virtual experiments)





$\frac{d\sigma}{dt} \neq 0, \frac{d\varepsilon}{dt} = 0, \sigma < Y$ → $\frac{d\varepsilon}{dt} = 0, \sigma < Y$

$\frac{d\sigma}{dt} = 0, \frac{d\varepsilon}{dt} \neq 0, \sigma = Y$ → $\frac{d\varepsilon}{dt} = c \ , \sigma = Y$

$\frac{d\sigma}{dt} \neq 0, \frac{d\varepsilon}{dt} = 0, \sigma < Y$ → $\frac{d\varepsilon}{dt} = 0, \sigma < Y$

Determine if $\sigma = Y$ is met, or not → yield criterion

H: Heaviside function;

$$H(x) = \begin{cases} 1, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

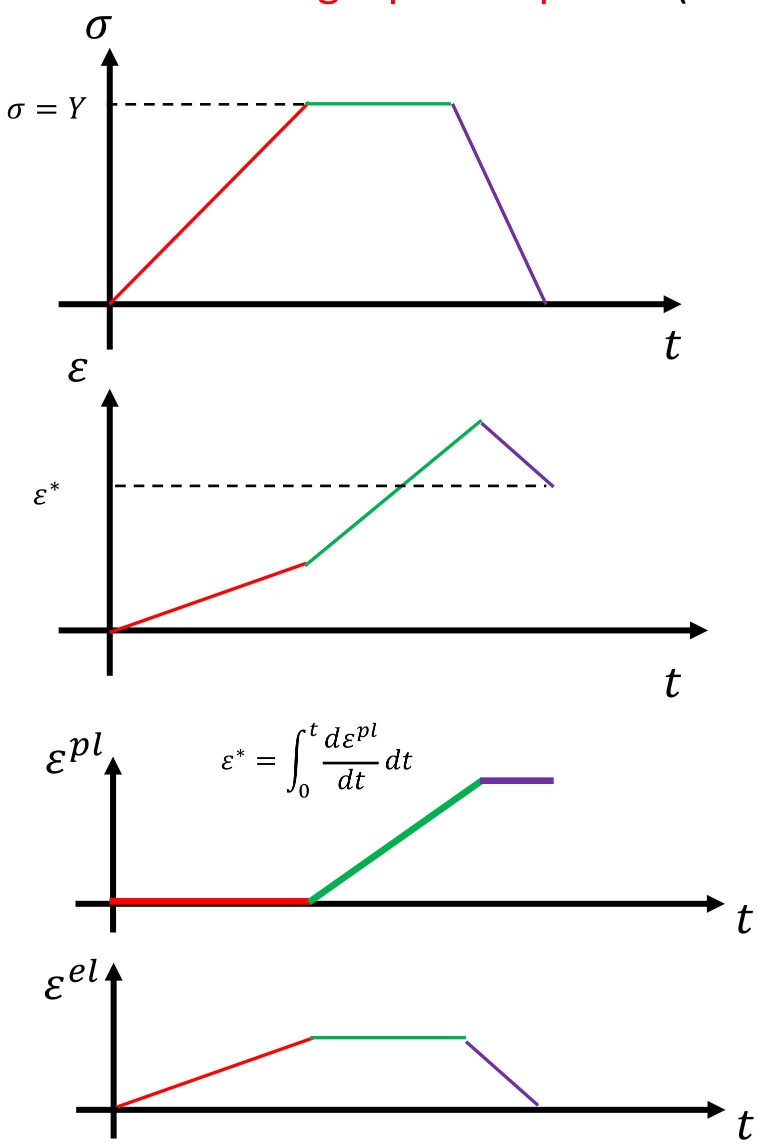
Similarly, Let's define $\hat{H}$

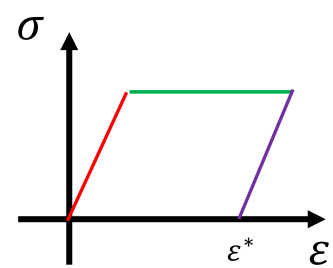
$$\hat{H}(\sigma - Y) \begin{cases} 1, \sigma = Y \\ 0, \sigma < Y \end{cases}$$

Constitutive model

$$\frac{d\varepsilon}{dt} = c\hat{H}(\sigma - Y)$$

Elastic-Rigid perfect plastic (virtual experiments)





$$\frac{d\sigma}{dt} \neq 0, \frac{d\varepsilon}{dt} \neq 0, \left(\frac{d\sigma}{dt} = E\frac{d\varepsilon}{dt}\right) \sigma < Y$$

$$\frac{d\sigma}{dt} \neq 0, \frac{d\varepsilon}{dt} \neq 0, \left(\frac{d\sigma}{dt} = E\frac{d\varepsilon}{dt}\right) \sigma < Y$$

$$\frac{d\varepsilon}{dt} = \frac{1}{E}\frac{d\sigma}{dt}, \sigma = E\varepsilon, \sigma < Y$$

$$\frac{d\sigma}{dt} = 0, \frac{d\varepsilon}{dt} \neq 0, \sigma = Y \qquad \frac{d\varepsilon}{dt} = c, \sigma = Y$$

$$\frac{d\varepsilon}{dt} = \frac{d\varepsilon^{pl}}{dt} + \frac{d\varepsilon^{el}}{dt}$$

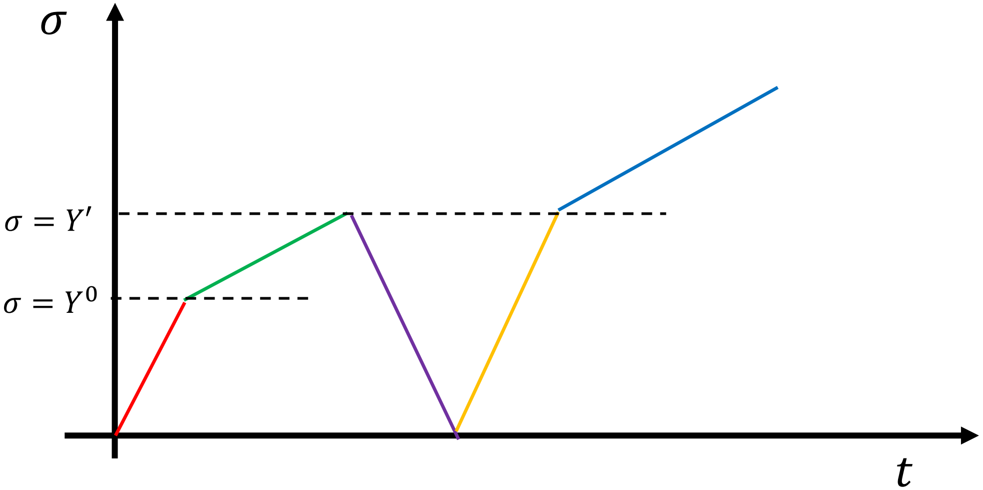
Maxwell assumption

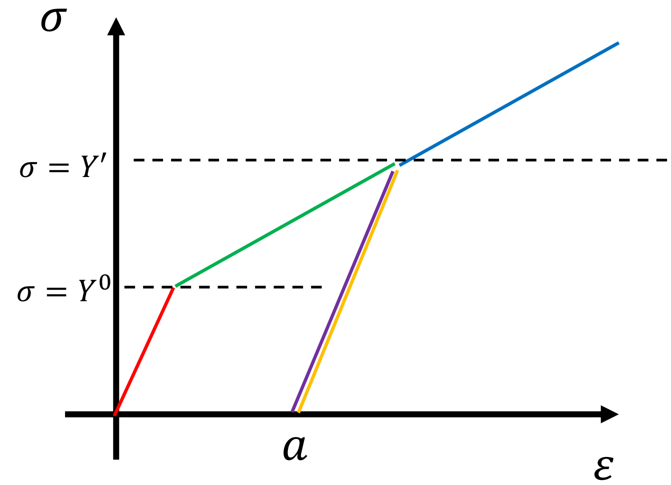
$$\frac{d\varepsilon}{dt} = c\hat{H}(\sigma - Y) + \frac{1}{E}\frac{d\sigma}{dt}$$

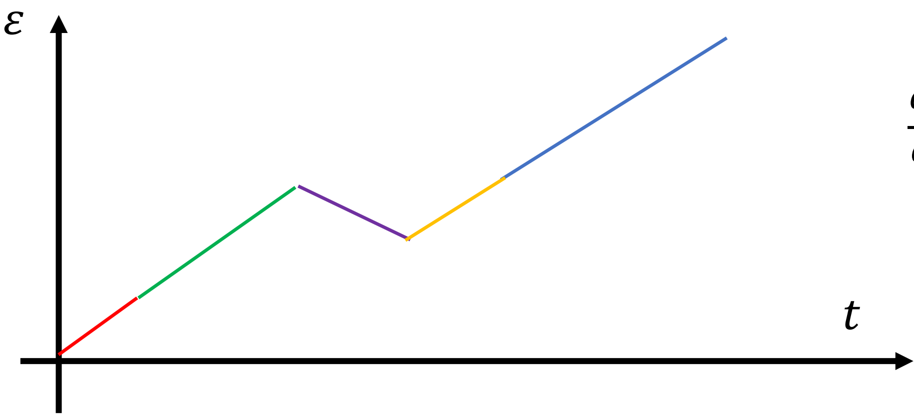
If $\sigma < Y$, $\quad \frac{d\varepsilon}{dt} = \frac{1}{E}\frac{d\sigma}{dt}$

If $\sigma = Y$, $\quad \frac{d\varepsilon}{dt} = c + \frac{1}{E}\frac{d\sigma}{dt}$

Elastic-plastic with linear hardening (virtual experiments)



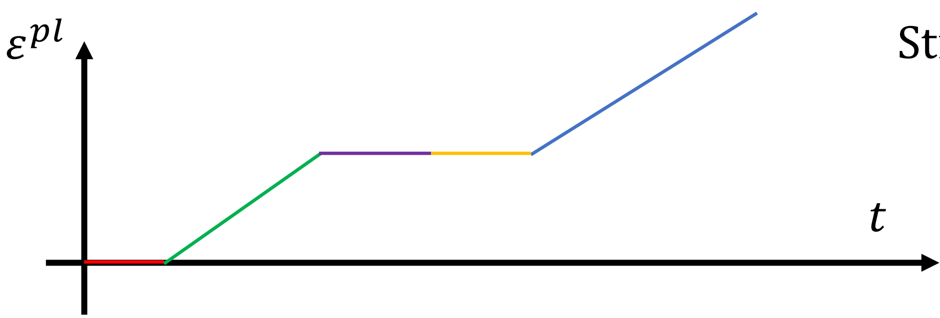




$$\frac{d\varepsilon}{dt} = c\hat{H}\left(\sigma - Y^0 - \frac{Y' - Y^0}{a}\varepsilon^{pl}(t)\right) + \frac{1}{E}\frac{d\sigma}{dt}$$

$$\frac{d\varepsilon}{dt} = c\hat{H}\left\{\sigma - Y^0 - \frac{Y' - Y^0}{a}\left(\varepsilon(t) - \varepsilon^{el}(t)\right)\right\} + \frac{1}{E}\frac{d\sigma}{dt}$$

$$\frac{d\varepsilon}{dt} = c\hat{H}\left\{\sigma - Y^0 - \frac{Y' - Y^0}{a}\left(\varepsilon(t) - \int_0^t \frac{1}{E}\frac{d\sigma}{dt}dt\right)\right\} + \frac{1}{E}\frac{d\sigma}{dt}$$



Strain-hardening:

$$Y = Y\left(\varepsilon^{pl}\right) \quad \rightarrow Y = Y^0 + \frac{Y' - Y^0}{a}\varepsilon^{pl}$$

$$\frac{d\varepsilon}{dt} = c\hat{H}\left\{\sigma - Y^0 - k\left(\varepsilon(t) - \int_0^t \frac{1}{E}\frac{d\sigma}{dt}dt\right)\right\} + \frac{1}{E}\frac{d\sigma}{dt}$$

Simulation using Google spread sheet

You'll need to create your own (custom) function to use $\widehat{H}$.

This site tells you how to create a custom function
https://developers.google.com/apps-script/guides/sheets/functions

I wrote my own as below:

```
function hhat(sig,y) {
  if (sig<y){
    return 0;
  }
  if (sig==y){
    return 1;
  }
  if (sig>y){
    return -1;
  }
}
```

Problem 1)

- Tensile a rigid-plastic body with the initial length of 10 in the rate of

$$\frac{dl}{dt} = 0.001 \left[\frac{mm}{sec}\right]$$

for 10 seconds.

- The body has the elastic modulus of 200,000 MPa and Y as 150 MPa.

To solve elasto-plastic problem with tensors…

- You'll need a special algorithm
- A most common (and basic) algorithm is 'return-mapping':
  1. At first, assume that the strain increment ($\Delta\varepsilon$) is purely elastic (i.e., $\Delta\varepsilon^{el} = \Delta\varepsilon$), and use Hooke's law to obtain stress increment ($\Delta\sigma = \mathrm{E}\Delta\varepsilon^{el}$).
  2. If stress + stress increment ($\sigma + \Delta\sigma$) does not meet yield criterion (i.e., $\phi(\sigma + \Delta\sigma) < Y$), you are safe with this assumption and you are all set. (fin.)
  3. If stress + stress increment ($\sigma + \Delta\sigma$) gives stress level that is higher than 'yield surface' ($\mathrm{i.e.}, \phi(\sigma + \Delta\sigma) > Y$), your assumption is wrong and iteratively reestimate elastic strain increment (thus plastic strain increment since $\Delta\varepsilon = \Delta\varepsilon^{pl} + \Delta\varepsilon^{el}$) until the condition $\phi(\sigma + \Delta\sigma) = Y$ is satisfied.
     - For this one, you'll need a numerical iteration (such as Newton-Raphson).
- There can be different algorithms which can replace the above algorithm. (Multiple solutions for a question)

Yield criterion with stress tensor (not scalar)

- In order to use tensorial quantities and apply the aforementioned method, we'll need to adjust a few assumptions.
- The use of Heaviside like function for yield criterion

$$\widehat{H}(\sigma - Y) \rightarrow \widehat{H}(\boldsymbol{\sigma}, Y). \qquad \widehat{H}\left(\sigma_{ij}, Y\right).$$

We cannot just subtract $\sigma_{ij} - Y$: note that σ_{ij} is a tensor, while Y is a scalar quantity.

We introduce a scalar function, called the yield function.
Yield criterion is described as a function of σ_{ij} (two free indices)

$\phi = \phi\left(\sigma_{ij}\right)$ and Let's use ϕ to see if yield condition is met ($\phi = Y$) or not ($\phi < Y$).

Strain-hardening with stress, strain tensors (not scalars)

- One use length of vector to quantify the 'size' of a vector.
- Similarly, we use equivalent scalar quantities as a (sort of) measure for sizes pertaining to stress and strain tensors.

- The equivalent scalar quantity for stress tensor is simply called 'equivalent stress', and the same is applied to strain tensor ('equivalent strain').

- There are a few types of equivalent quantities. We'll use only von Mises quantity.

$$\bar{\sigma}^{eq} = \left[\frac{1}{2}\{(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2)\}\right]^{0.5}$$

- The definition of equivalent strain is not straightforward. We'll have to determine the amount of plastic work done to the material that is under the given stress σ_{ij} (thus providing us the above equivalent stress tensor).

Plastic work done

- Apparently, plasticity is non-linear, the work done for the time from 0 to t is defined using integration:
- $w^{pl}(t) = w^{pl}(t=0) + \int_0^t dw$
- $dw^{pl} = \boldsymbol{\sigma}: d\boldsymbol{\varepsilon}^{pl} = \sigma_{ij} d\varepsilon_{ij}^{pl}$ (two dummy indices i, j)
- We postulate the work done calculated by stress and strain tensor (as done above) should be the same as the one calculated by the equivalent stress, equivalent strain.
- The above postulation is expressed as below:
- $dw^{pl} = \sigma_{ij} d\varepsilon_{ij}^{pl} = \sigma^{eq} d\varepsilon^{eq} \rightarrow d\varepsilon^{eq} = \frac{\sigma_{ij} d\varepsilon_{ij}^{pl}}{d\sigma^{eq}}$

$$\varepsilon^{eq}(t) = \varepsilon^{eq}(t=0) + \int_0^t d\varepsilon^{eq} = \varepsilon^{eq}(t=0) + \int_0^t \frac{\sigma_{ij} d\varepsilon_{ij}^{pl}}{d\sigma^{eq}}$$

- For materials without previous deformation, the above can be written as:

$$\varepsilon^{eq}(t) = \int_0^t \frac{\boldsymbol{\sigma}: d\boldsymbol{\varepsilon}^{pl}}{d\sigma^{eq}}$$

Strain hardening?

- Say, your material obeys the Hollomon equation,

$$\sigma = Y^0 + K'\varepsilon^n$$

$\boldsymbol{\sigma} = Y^0 + K'\boldsymbol{\varepsilon}^n$ (not possibly); The mathematical operation of 'exponential' is not available.

$\sigma^{eq} = Y^0 + K'(\varepsilon^{eq})^n$; Instead of using the tensors, we use 'equivalent' quantities (that are sca

- In case your material obeys linear hardening:

$$\sigma^{eq} = Y^0 + K\varepsilon^{eq}$$

Now, let's look at the constitutive model again

$$\frac{d\boldsymbol{\varepsilon}}{dt} = c\widehat{H}\left\{\boldsymbol{\sigma} - Y^0 - k\left(\boldsymbol{\varepsilon}(t) - \int_0^t \frac{1}{\mathbb{E}} : \frac{d\boldsymbol{\sigma}}{dt} dt\right)\right\} + \frac{1}{E} : \frac{d\boldsymbol{\sigma}}{dt}$$

Linear strain hardening: $\sigma^{eq} = K'\varepsilon^{eq}$ should somehow replace $\mathrm{Y}^0 + k\left(\varepsilon(t) - \int_0^t \frac{1}{E}\frac{d\sigma}{dt} dt\right)$;
Since yield stress increases as more plastic deformation is applied.

$$\sigma - Y^0 - k\left(\varepsilon(t) - \int_0^t \frac{1}{E}\frac{d\sigma}{dt} dt\right) \rightarrow \quad \phi(\boldsymbol{\sigma}) - Y^0 - \sigma^{eq}(\varepsilon^{eq}) \qquad \varepsilon^{eq}(t) = \int_0^t \frac{\boldsymbol{\sigma} : d\boldsymbol{\varepsilon}}{d\sigma^{eq}}$$

c should be somehow replaced with a tensorial quantity similar to $\frac{d\boldsymbol{\varepsilon}^{pl}}{dt}$, for now,
Let's replace c with $\frac{\mathrm{d}\lambda}{dt}\left(\boldsymbol{\varepsilon}^{pl}\right)$ and $d\lambda$ is differential form of a scalar quantity that is a function of plastic strain $\boldsymbol{\varepsilon}^{pl}$.

Now, let's look at the constitutive model again

$$\frac{d\boldsymbol{\varepsilon}}{dt} = c\hat{H}\left\{\boldsymbol{\sigma} - Y^0 - k\left(\boldsymbol{\varepsilon}(t) - \int_0^t \frac{1}{\mathbb{E}}:\frac{d\boldsymbol{\sigma}}{dt}dt\right)\right\} + \frac{1}{\mathbb{E}}:\frac{d\boldsymbol{\sigma}}{dt} \qquad \frac{d\boldsymbol{\varepsilon}}{dt} = \frac{d\boldsymbol{\varepsilon}^{pl}}{dt}\hat{H}(\phi(\boldsymbol{\sigma}) - Y^0 - K'\varepsilon^{eq}) + \frac{1}{\mathbb{E}}:\frac{d\boldsymbol{\sigma}}{dt}$$

Linear strain hardening: $\sigma^{eq} = Y^0 + K'\varepsilon^{eq}$ should somehow replace $\mathrm{Y}^0 + k\left(\boldsymbol{\varepsilon}(t) - \int_0^t \frac{1}{\mathbb{E}}:\frac{d\boldsymbol{\sigma}}{dt}dt\right)$; Since yield stress increases as more plastic deformation is applied.

$$\boldsymbol{\sigma} - Y^0 - k\left(\boldsymbol{\varepsilon}(t) - \int_0^t \frac{1}{\mathbb{E}}:\frac{d\boldsymbol{\sigma}}{dt}dt\right) \rightarrow \quad \phi(\boldsymbol{\sigma}) - Y^0 - \sigma^{eq}(\varepsilon^{eq}) \qquad \varepsilon^{eq}(t) = \int_0^t \frac{\boldsymbol{\sigma}:d\boldsymbol{\varepsilon}}{d\sigma^{eq}}$$

c should be somehow replaced with a tensorial quantity similar to $\frac{d\boldsymbol{\varepsilon}^{pl}}{dt}$, for now,
Let's replace c with $\frac{d\boldsymbol{\varepsilon}^{pl}}{dt}$ as $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{pl} + \boldsymbol{\varepsilon}^{el}$. ($\rightarrow \frac{d\boldsymbol{\varepsilon}}{dt} = \frac{d\boldsymbol{\varepsilon}^{pl}}{dt} + \frac{d\boldsymbol{\varepsilon}^{el}}{dt}$)

Finding unknown plastic strain

$$\frac{d\boldsymbol{\varepsilon}}{dt} = \boxed{\frac{d\boldsymbol{\varepsilon}^{pl}}{dt}\widehat{H}\left(\phi(\boldsymbol{\sigma}) - Y^0 - K'\varepsilon^{eq}\right)} + \frac{1}{\mathbb{E}} : \frac{d\boldsymbol{\sigma}}{dt}$$

$$\frac{d\boldsymbol{\varepsilon}}{dt} = \frac{d\boldsymbol{\varepsilon}^{pl}}{dt}\widehat{H}\left(\phi(\boldsymbol{\sigma}) - Y^0 - K'\varepsilon^{eq}\right) + \frac{1}{\mathbb{E}} : \frac{d\boldsymbol{\sigma}}{dt} \qquad \frac{d\boldsymbol{\varepsilon}}{dt} = \frac{d\boldsymbol{\varepsilon}^{pl}}{dt} + \frac{1}{\mathbb{E}} : \frac{d\boldsymbol{\sigma}}{dt}$$

$$\frac{d\boldsymbol{\varepsilon}^{pl}}{dt} ??$$

Let's postulate Hill's idea (associated flow rule)

$$\frac{d\boldsymbol{\varepsilon}^{pl}}{dt} = \frac{d\varepsilon^{eq}}{dt}\left(\frac{\partial\phi(\boldsymbol{\sigma})}{\partial\boldsymbol{\sigma}}\right)$$

If we use von Mises yield criterion:

$$\frac{d\varepsilon_{ij}^{pl}}{dt} = \frac{d\varepsilon^{eq}}{dt}\left(\frac{\partial\left\{\left[\frac{1}{2}\{(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2)\}\right]^{0.5}\right\}}{\partial\sigma_{ij}}\right)$$

If we use von Mises yield criterion:

$$\frac{d\varepsilon_{ij}^{pl}}{dt} = \frac{d\left\{\int_0^t \frac{\boldsymbol{\sigma}:d\boldsymbol{\varepsilon}}{d\sigma^{eq}}\right\}}{dt}\left(\frac{\partial\left\{\left[\frac{1}{2}\{(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2)\}\right]^{0.5}\right\}}{\partial\sigma_{ij}}\right)$$

Yet, another example with scalars

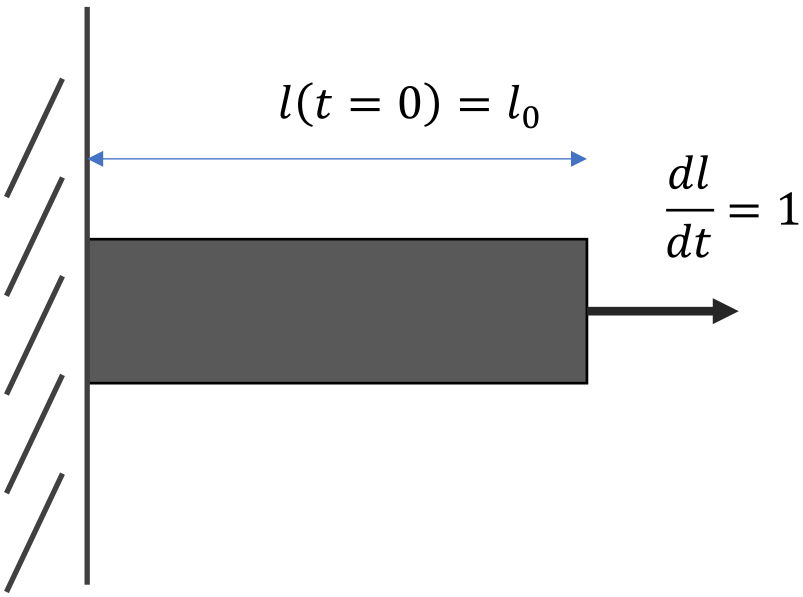
If you are not yet familiar with vector (tensor) algebra and numerical analysis, just feel how much mathematics are involved …

Problem: stretching an elasto-perfect-plastic rod

$$\frac{d\boldsymbol{\varepsilon}}{dt} = \frac{d\boldsymbol{\varepsilon}^{pl}}{dt}\widehat{H}(\phi(\boldsymbol{\sigma}) - Y^0) + \frac{1}{\mathbb{E}}:\frac{d\boldsymbol{\sigma}}{dt}$$

Perfect-plastic (no hardening; Y^0 is constant)

with associated flow rule: $\frac{d\varepsilon^{eq}}{dt}\left(\frac{\partial\phi(\boldsymbol{\sigma})}{\partial\boldsymbol{\sigma}}\right)$



For the case of the current 'scalar' example,

$$\varepsilon^{eq} = \varepsilon, \sigma^{eq} = \sigma$$

Let's say, your yield function $\phi(\sigma)$ is

$$\phi(\sigma) = \sqrt{\sigma^2}$$

Your yield criterion:

$$\phi(\sigma) = \sqrt{\sigma^2} = 100$$

Say, your material yield property, $Y^0 = 100$

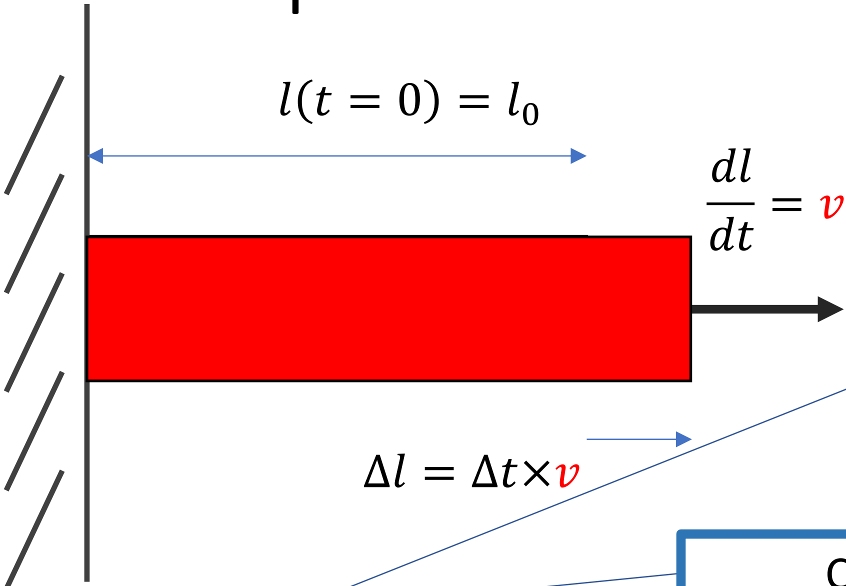
Your yield function gives

$$\frac{d\phi(\sigma)}{d\sigma} = 1$$

and

$\phi(\sigma = 100) = 100$ satisfies yield condition

Elastic predictor and corrector algorithm



While time increment is Δt

$$\Delta\varepsilon = \frac{\Delta l}{l}$$

Initially assuming that all strain is elastic

$$\Delta\sigma = \mathbb{E}\Delta\varepsilon = \mathbb{E}\frac{\Delta l}{l}$$

Stress is updated:

$$\sigma_{(n+1)} = \sigma_{(n)} + \Delta\sigma$$

$$\sigma_{(n+1)} = \sigma_{(n)} + \mathbb{E}\Delta\varepsilon = \sigma_{(n)} + \mathbb{E}\frac{\Delta l}{l}$$

Check if $\sigma_{(n+1)}$ really gives only elastic contribution

$\phi(\sigma_{(n+1)}) < c$ or $\phi(\sigma_{(n+1)}) = c$ or $\phi(\sigma_{(n+1)}) > c$

You are correct about the strain decomposition, let's move on to next time increment (end)

Plasticity update

$$\Delta\varepsilon^{pl} = \Delta\varepsilon^{eq}\frac{\partial\phi}{\partial\sigma}$$

$$\Delta\varepsilon = \Delta\varepsilon^{el} + \Delta\varepsilon^{pl}$$

Correcting elastic strain $\Delta\sigma = \mathbb{E}(\Delta\varepsilon - \Delta\varepsilon^{pl})$

Elastic predictor and corrector algorithm

$$\Delta l = \Delta t \times v$$

Initially assuming that all strain is elastic

$$\Delta\sigma = \mathbb{E}\Delta\varepsilon = \mathbb{E}\frac{\Delta l}{l}$$

Find the stress corresponding to the current stress increment

$$\sigma_{(n+1)} = \sigma_{(n)} + \Delta\sigma$$

Check if $\sigma_{(n+1)}$ is inside, on or over the yield criterion

$\phi\left(\sigma_{(n+1)}\right) < Y^0$ or $\phi\left(\sigma_{(n+1)}\right) = Y^0$ or $\phi\left(\sigma_{(n+1)}\right) > Y^0$

If $\phi\left(\sigma_{(n+1)}\right) \leq Y^0$

$\sigma_{(n+1)}$ is consistent with our theory.

Let's move on to next time increment

Plastic strain update

$$\Delta\varepsilon^{pl} = \Delta\varepsilon^{eq}\frac{\partial\phi}{\partial\sigma}$$

Correcting elastic strain

$$\Delta\varepsilon^{el} = \Delta\varepsilon - \Delta\varepsilon^{pl}$$

That gives new stress increment

$$\Delta\sigma = \mathbb{E}(\Delta\varepsilon - \Delta\varepsilon^{pl})$$

(adjust strain decomposition

Plastic strain update

$$\Delta\varepsilon^{pl} = \Delta\varepsilon^{eq}\frac{\partial\phi}{\partial\sigma}$$

$F(\Delta\lambda) = \phi(\Delta\varepsilon^{eq}) - (Y^0)$, find $\Delta\varepsilon^{eq}$ that gives $F(\Delta\varepsilon^{eq}) = 0$

That's exactly what NR can do.

$$\frac{dF(\Delta\varepsilon^{eq})}{d\Delta\varepsilon^{eq}} = \frac{d(\phi(\sigma) - Y^0)}{d\Delta\varepsilon^{eq}} = \frac{d\phi(\sigma)}{d\Delta\varepsilon^{eq}} = \frac{d\phi(\sigma)}{d\sigma}\frac{d\sigma}{d\Delta\varepsilon^{eq}} = 1\times\frac{d(\sigma_{(n)} + \Delta\sigma)}{d\Delta\varepsilon^{eq}} = \frac{d\Delta\sigma}{d\Delta\varepsilon^{eq}}$$

$$= \frac{d\mathbb{E}\left(\Delta\varepsilon - \Delta\varepsilon^{pl}\right)}{d\Delta\varepsilon^{eq}} \approx \mathbb{E}\frac{d\Delta\varepsilon^{pl}}{d\Delta\varepsilon^{eq}} = \mathbb{E}\,\mathrm{sgn}\left(\Delta\varepsilon^{pl}\right)$$

$$\Delta\varepsilon^{eq}_{(k+1)} = \Delta\varepsilon^{eq}_{(k)} - \frac{F\left(\Delta\varepsilon^{eq}_{(k)}\right)}{-\sigma\mathbb{E}\,\mathrm{sgn}(\Delta\varepsilon^{pl})}$$

$\mathrm{sgn}\left(\Delta\varepsilon^{pl}\right) = 1$ if $\Delta\varepsilon^{pl} \geq 0$

$\mathrm{sgn}\left(\Delta\varepsilon^{pl}\right) = -1$ if $\Delta\varepsilon^{pl} < 0$

Caution! Need validation

Cheat sheet

```fortran
      real function calc_yield_function(stress)
      implicit none
      real stress
      calc_yield_function = sqrt(stress**2.)
      return
      end function
```

```fortran
      program elasto_plasticity_scalar
      implicit none
      real calc_yield_function
      real dt, E, c, t, l, dl, eps, deps, deps_el, deps_pl,dsig,
     $     stress
      real vel, f, tol
      integer kount,iplast
      parameter(tol=1e-6)

      open(3,file='elasto_plasticity_scalar.txt')
      dt = 0.02                    ! time increment
      E = 200000                   ! elastic modulus
      c = 200.                     ! yield criterion
      stress = 0.                  ! initial stress
      eps = 0.                     ! initial strain
      l = 1.                       ! initial length
      t = 0.                       ! initial time
```

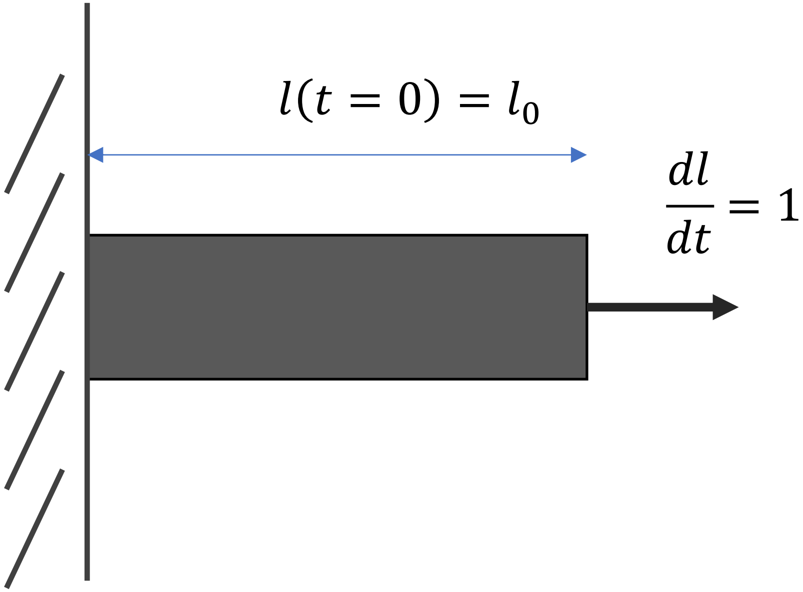
```fortran
      do while(t<1.0)
c     Loading condition 1
         if (t.le.0.25) then
            vel = 0.01
         elseif (t.gt.0.25.and.t.le.0.55) then
            vel =-0.01
         else
            vel = 0.01
         endif
         dl = vel * dt
         deps = dl / l
c        initially assuming all strain is elastic
         deps_pl = 0.0
         deps_el = deps
c        guess on stress increment
         dsig = E * deps_el
         kount = 0
         f = calc_yield_function( stress+ dsig) - c
         iplast=0
         do while (f.gt.tol.and.kount.lt.3) ! if exceeding plastic onset
            iplast=1
            deps_el = deps - deps_pl
            dsig = e * deps_el
            f = calc_yield_function(stress+dsig) - c
c           estimate new plastic increment
            deps_pl = deps_pl - f/(-E)*sign(1.,deps)
            kount = kount +1
         enddo
         write(3,'(2f9.4,2f10.4,2i2)')t,l,eps,stress,iplast,kount
         stress = stress + dsig
         eps = eps + deps
         t = t + dt
         l = l + dl
      enddo
      close(3)
      end program
```

Problem: stretching an elasto-plastic rod with linear hardening

$$\frac{d\boldsymbol{\varepsilon}}{dt} = \frac{d\boldsymbol{\varepsilon}^{pl}}{dt}\widehat{H}(\phi(\boldsymbol{\sigma}) - (Y^0 + k\varepsilon^{eq})) + \frac{1}{\mathbb{E}}:\frac{d\boldsymbol{\sigma}}{dt}$$

Perfect-plastic (no hardening; Y^0 is constant)

with associated flow rule: $\frac{d\varepsilon^{eq}}{dt}\left(\frac{\partial\phi(\boldsymbol{\sigma})}{\partial\boldsymbol{\sigma}}\right)$



For the case of the current 'scalar' example,

$$\varepsilon^{eq} = \varepsilon, \sigma^{eq} = \sigma$$

Let's say, your yield function $\phi(\sigma)$ is

$$\phi(\sigma) = \sqrt{\sigma^2}$$

Your yield criterion:

$$\phi(\sigma) = \sqrt{\sigma^2} = 100$$

Say, your material yield property, Y^0

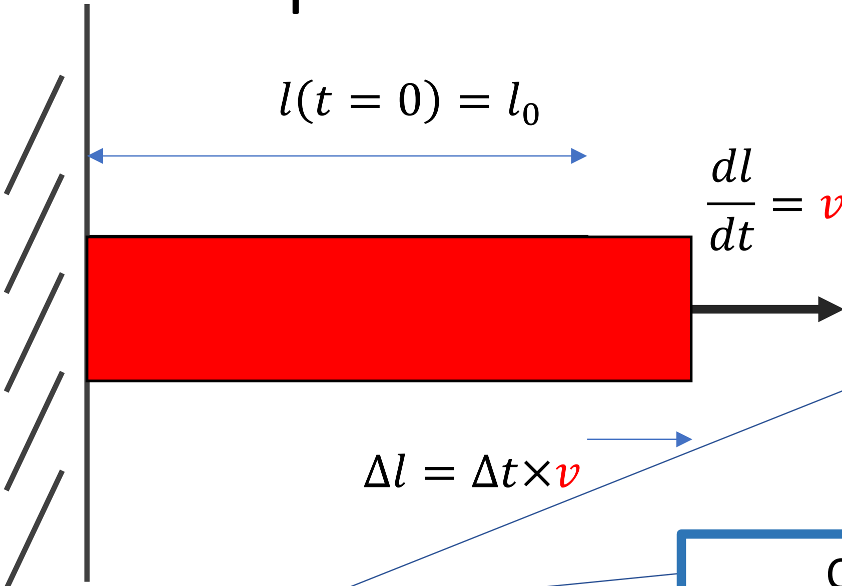
Your yield function gives

$$\frac{d\phi(\sigma)}{d\sigma} = 1$$

and

$\phi(\sigma = 100) = 100$ satisfies yield condition

Elastic predictor and corrector algorithm



$l(t=0) = l_0$

$\frac{dl}{dt} = v$

$\Delta l = \Delta t \times v$

While time increment is Δt

$$\Delta\varepsilon = \frac{\Delta l}{l}$$

Initially assuming that all strain is elastic

$$\Delta\sigma = \mathbb{E}\Delta\varepsilon = \mathbb{E}\frac{\Delta l}{l}$$

Correcting elastic strain $\Delta\sigma = \mathbb{E}(\Delta\varepsilon - \Delta\varepsilon^{pl})$

Stress is updated:

$$\sigma_{(n+1)} = \sigma_{(n)} + \Delta\sigma$$

$$\sigma_{(n+1)} = \sigma_{(n)} + \mathbb{E}\Delta\varepsilon$$

$$= \sigma_{(n)} + \mathbb{E}\frac{\Delta l}{l}$$

Check if $\sigma_{(n+1)}$ really gives only elastic contribution

$\phi(\sigma_{(n+1)}) < c$ or $\phi(\sigma_{(n+1)}) = c$ or $\phi(\sigma_{(n+1)}) > c$

You are correct about the strain decomposition, let's move on to next time increment (end)

Plasticity update

$$\Delta\varepsilon^{pl} = \Delta\varepsilon^{eq}\frac{\partial\phi}{\partial\sigma}$$

$$\Delta\varepsilon = \Delta\varepsilon^{el} + \Delta\varepsilon^{pl}$$

Elastic predictor and corrector algorithm

$$\Delta l = \Delta t \times v$$

Initially assuming that all strain is elastic

$$\Delta\sigma = \mathbb{E}\Delta\varepsilon = \mathbb{E}\frac{\Delta l}{l}$$

Find the stress corresponding to the current stress increment

$$\sigma_{(n+1)} = \sigma_{(n)} + \Delta\sigma$$

Correcting elastic strain

$$\Delta\varepsilon^{el} = \Delta\varepsilon - \Delta\varepsilon^{pl}$$

That gives new stress increment

$$\Delta\sigma = \mathbb{E}(\Delta\varepsilon - \Delta\varepsilon^{pl})$$

(adjust strain decomposition

Check if $\sigma_{(n+1)}$ is inside, on or over the yield criterion

$$\phi\left(\sigma_{(n+1)}\right) < Y^0 + k(\varepsilon_{(n)}^{eq} + \Delta\varepsilon^{eq})$$

or $\phi\left(\sigma_{(n+1)}\right) = Y^0 + k(\varepsilon_{(n)}^{eq} + \Delta\varepsilon^{eq})$

or $\phi\left(\sigma_{(n+1)}\right) > Y^0 + k(\varepsilon_{(n)}^{eq} + \Delta\varepsilon^{eq})$

Plastic strain update

$$\Delta\varepsilon^{pl} = \Delta\varepsilon^{eq}\frac{\partial\phi}{\partial\sigma}$$

If $\phi\left(\sigma_{(n+1)}\right) \leq Y^0 + k(\varepsilon_{(n)}^{eq} + \Delta\varepsilon^{eq})$

$\sigma_{(n+1)}$ is consistent with our theory.

Let's move on to next time increment

Plastic strain update

$$\Delta\varepsilon^{pl} = \Delta\varepsilon^{eq}\frac{\partial\phi}{\partial\sigma}$$

That's exactly what NR can do.

$F(\Delta\lambda) = \phi(\Delta\varepsilon^{eq}) - \left(Y^0 + k\left(\varepsilon^{eq}_{(n)} + \Delta\varepsilon^{eq}\right)\right)$, find $\Delta\varepsilon^{eq}$ that gives $F(\Delta\varepsilon^{eq}) = 0$

$$\frac{dF(\Delta\varepsilon^{eq})}{d\Delta\varepsilon^{eq}} = \frac{d\left(\phi(\sigma) - \left(Y^0 + k\left(\varepsilon^{eq}_{(n)} + \Delta\varepsilon^{eq}\right)\right)\right)}{d\Delta\varepsilon^{eq}} = \frac{d\phi(\sigma)}{d\Delta\varepsilon^{eq}} - k = \frac{d\phi(\sigma)}{d\sigma}\frac{d\sigma}{d\Delta\varepsilon^{eq}} - k$$

$$= 1\times\frac{d(\sigma_{(n)} + \Delta\sigma)}{d\Delta\varepsilon^{eq}} - k = \frac{d\Delta\sigma}{d\Delta\varepsilon^{eq}} - k = \frac{d\mathbb{E}\left(\Delta\varepsilon - \Delta\varepsilon^{pl}\right)}{d\Delta\varepsilon^{eq}} - k \approx \mathbb{E}\frac{d\Delta\varepsilon^{pl}}{d\Delta\varepsilon^{eq}} - k = \mathbb{E}\,\mathrm{sgn}\left(\Delta\varepsilon^{pl}\right) - k$$

$$\Delta\varepsilon^{eq}_{(k+1)} = \Delta\varepsilon^{eq}_{(k)} - \frac{F\left(\Delta\varepsilon^{eq}_{(k)}\right)}{-\sigma\mathbb{E}\,\mathrm{sgn}(\Delta\varepsilon^{pl}) + k}$$

$\mathrm{sgn}\left(\Delta\varepsilon^{pl}\right) = 1$ if $\Delta\varepsilon^{pl} \geq 0$

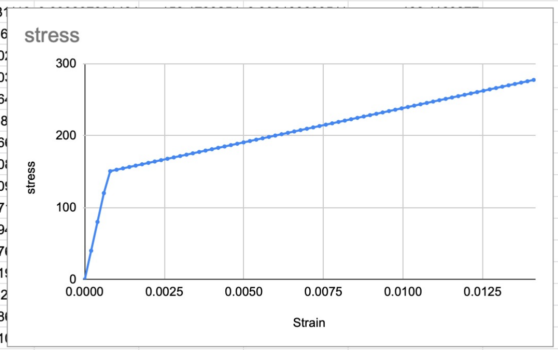
$\mathrm{sgn}\left(\Delta\varepsilon^{pl}\right) = -1$ if $\Delta\varepsilon^{pl} < 0$

Caution! Need validation

Google spreadsheet example on elasticity-linear plasticity (scalar)

https://docs.google.com/spreadsheets/d/1ithHT-f_mQIyv9Vx2fBt_o6bMWRq8Hlb0N5VXrw2ySA/edit?usp=sharing

my loading condition									Elast Modulus (M
time incr (dt)	0.2		dl/dt	0.01	Delta l	0.002			200,000
time	length	eps	eps_pl	stress	dstrain	dsigma (pure)+stress	H (sig^guss-Y)	deps_elasticity	deps_plas
0	10	0	0	0	0.0002	40	0	0.0002	0
0.2	10.002	0.0002	0	40	0.000199960008	79.9920016	0	0.000199960008	0
0.4	10.004	0.000399960008	0	79.9920016	0.000199920032	119.976008	0	0.000199920032	0
0.6	10.006	0.00059988004	0	119.976008	0.000199880072	159.9520224	-1	0.000152607965	0.000047272106
0.8	10.008	0.00079976011	0.000047272106	150.4976011	0.000199840127	190.4656267	-1	0.000009873826	0.000189966301
1	10.01	0.000999600239	0.000237238408	152.4723663	0.000199800199	192.4324063	-1	0.000009515094	0.000190285105
1.2	10.012	0.00119940044	0.000427523513	154.3753852	0.000199760287	194.3274427	-1	0.000009512301	0.000190247986
1.4	10.014	0.001399160727	0.000617771499	156.2778455	0.000199720391	196.2219238	-1	0.000009510399	0.000190209991
1.6	10.016	0.00159888	[illegible]	[illegible]	[illegible]	[illegible]	-1	0.000009508500	0.000190172010
1.8	10.018	0.00179856	[illegible]	[illegible]	[illegible]	[illegible]	-1	0.000009506602	0.000190134044
2	10.02	0.00199820	[illegible]	[illegible]	[illegible]	[illegible]	-1	0.000009504704	0.000190096093
2.2	10.022	0.00219780	[illegible]	[illegible]	[illegible]	[illegible]	-1	0.000009502808	0.000190058157
2.4	10.024	0.00239736	[illegible]	[illegible]	[illegible]	[illegible]	-1	0.000009500912	0.000190020237
2.6	10.026	0.00259688	[illegible]	[illegible]	[illegible]	[illegible]	-1	0.000009499016	0.000189982331
2.8	10.028	0.00279636	[illegible]	[illegible]	[illegible]	[illegible]	-1	0.000009497122	0.000189944441
3	10.03	0.00299580	[illegible]	[illegible]	[illegible]	[illegible]	-1	0.000009495228	0.000189906566
3.2	10.032	0.00319520	[illegible]	[illegible]	[illegible]	[illegible]	-1	0.000009493335	0.000189868705
3.4	10.034	0.00339457	[illegible]	[illegible]	[illegible]	[illegible]	-1	0.000009491443	0.000189830860
3.6	10.036	0.00359389	[illegible]	[illegible]	[illegible]	[illegible]	-1	0.000009489551	0.000189793030
3.8	10.038	0.00379317	[illegible]	[illegible]	[illegible]	[illegible]	-1	0.000009487661	0.000189755215
4	10.04	0.00399241	[illegible]	[illegible]	[illegible]	[illegible]	-1	0.000009485771	0.000189717415
4.2	10.042	0.00419162	[illegible]	[illegible]	[illegible]	[illegible]	-1	0.000009483882	0.000189679631
4.4	10.044	0.00439078	[illegible]	[illegible]	[illegible]	[illegible]	-1	0.000009481993	0.000189641861
4.6	10.046	0.00458991	[illegible]	[illegible]	[illegible]	[illegible]	-1	0.000009480105	0.000189604106
4.8	10.048	0.004788994471	0.003846186202	188.5616537	0.000199044586	228.3705709	-1	0.000009478219	0.000189566367




```
function decompose_lh(s,E,y,de,eeq,k){
// elasticity - plasticity with linear hardening
// initial guess with assuming pure elasticity
deq=0
del=de-deq
// F=yield(s+E*del)-y
tolerance=1e-9
F=2*tolerance
kount=0
while(F>tolerance){
// calculate F
F=yield(s+E*del)-(y+k*(eeq+deq))
deq=deq-F/(-E)//*sign(1,de)
if (de<0) {
deq=deq*-1
}
del=de-deq
kount=kount+1
if (kount>20){
throw 'too many iterations', kount
}
}
return del
}
```

Or, Jupyter notebook (Python)

In [1]:
```
%pylab inline
```

```
Populating the interactive namespace from numpy and matplotli
b
```

- Yield surface $\phi = \sqrt{\sigma^2}$

In [2]:
```
def ys(s): return np.sqrt(s**2)
```

In [3]:
```
## predictor corrector algorithm to determine plastic strain i
def decompose(s,E,y0,de,eeq,k):
    deq=0
    deel=de-deq
    tolerance=1e-9
    #F=2*tolerance
    F=ys(s+E*deel)-(y0+k*(eeq+deq))
    kount=0
    while F>tolerance:
        F=ys(s+E*deel)-(y0+k*(eeq+deq))
        deq=deq-F/((-E)*np.sign(de))
        deel=de-deq
        kount=kount+1
        #print('deq:',deq)
        if kount>20:
            raise IOError("something went wrong")
    return deel
```

물성

In [4]:
```
E=200000 ## 영률 (Young's modulus)
y0=200   ## 초기 항복 강도
k=10000  ## Hardening parameter
decompose(100,E,y0,1e-3,0,k)
```

Out[4]: `0.0005250000000000001`

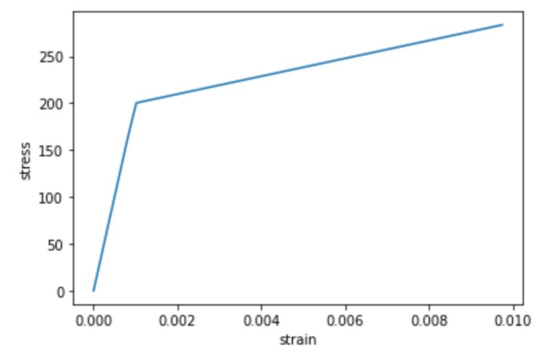
하중 조건 (일축인장)

In [5]:
```
t=np.linspace(0,10) ## 시간 0초 부터 10초 까지
l=np.linspace(10,10.1) ## 길이 10[mm]에서 10.1[mm]까지 변형
```

In [6]:
```
## initially zero stress, zero plastic equivalent strain
s=0
eeq=0
e=0
x=[]
y=[]
for i in range(len(t)-1):
    #x.append(eeq)
    x.append(e)
    y.append(s)
    dl=l[i+1]-l[i]
    de=dl/l[i]
    e=e+de
    deel = decompose(s,E,y0,de,eeq,k)
    s=s+deel*E
    depl=de-deel
    eeq=eeq+depl
```

In [7]:
```
fig=plt.figure();ax=fig.add_subplot(111)
ax.plot(x,y,'-')
ax.set_xlabel('strain')
ax.set_ylabel('stress')
```

Out[7]: `Text(0, 0.5, 'stress')`



https://gist.github.com/youngung/07615666a85d663eae9cbb157fa33e38