Elasticity

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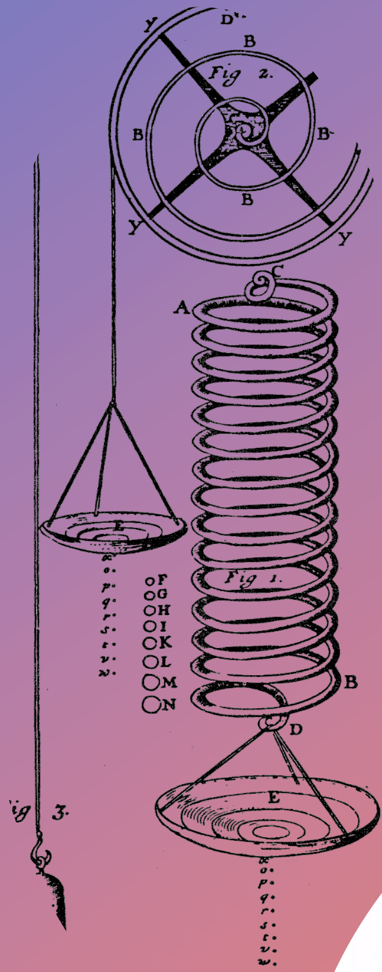
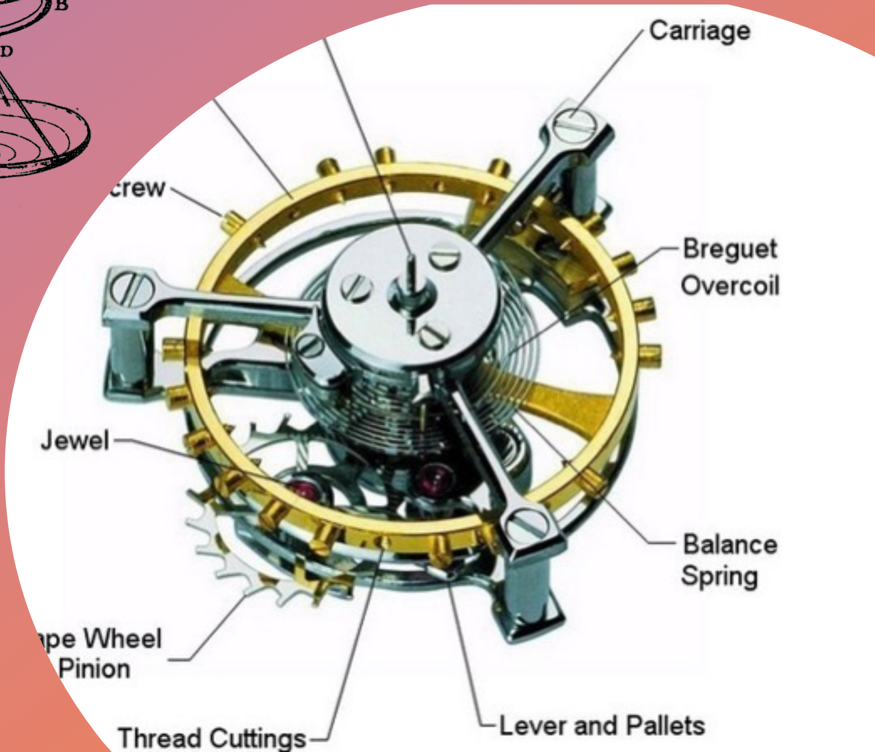
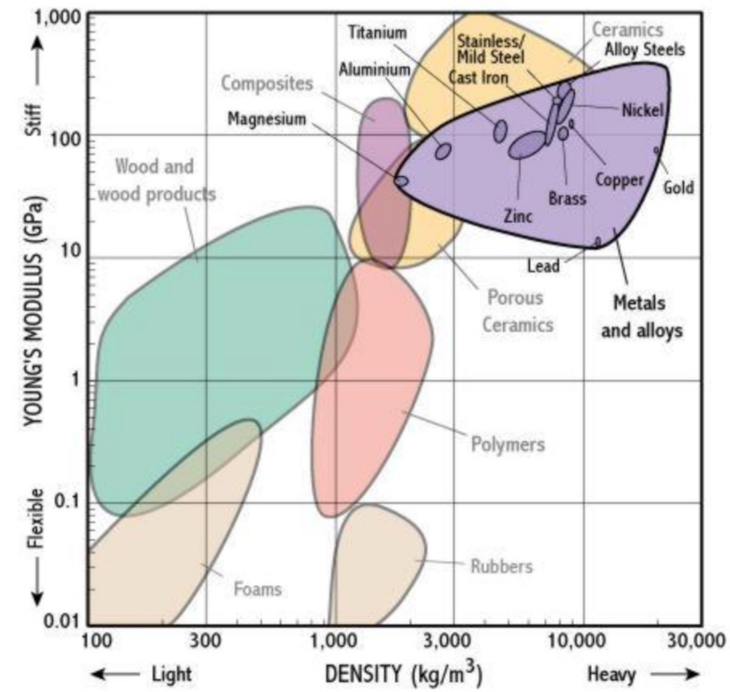




탄성 변형

- 기계적 자극과 기계적 반응, 그리고 그 둘의 관계 (간단한 예: 금속의 탄성)
  - $\sigma = \mathbb{E}\varepsilon$ (σ: 응력 $\mathbb{E}$: elastic constants (modulus), ε: 변형률)
- 위는 금속의 탄성 구간에 적용이 가능. $\mathbb{E}$ 값은 양수. 따라서, 음의 응력을 가하면 음의 변형률이 얻어진다. 양의 응력이 점점 커질수록 양의 변형률이 점점 커진다.
- 위의 관계식은 Hooke’s law로 불린다. 비례 상수 $\mathbb{E}$ 는 다양한 이름으로 불린다:
  - Elastic modulus
  - Elastic Constants
  - Young’s modulus 등
- Hooke’s law의 $\mathbb{E}$는 해당 물질의 탄성 구간에서의 물성 (material property)이다. 따라서 재료마다 상이한 값을 가진다.
- 교재 227쪽 탄성 변형에 대한 정의는 틀렸다. 이는 다음 장에서 더 알아보자.

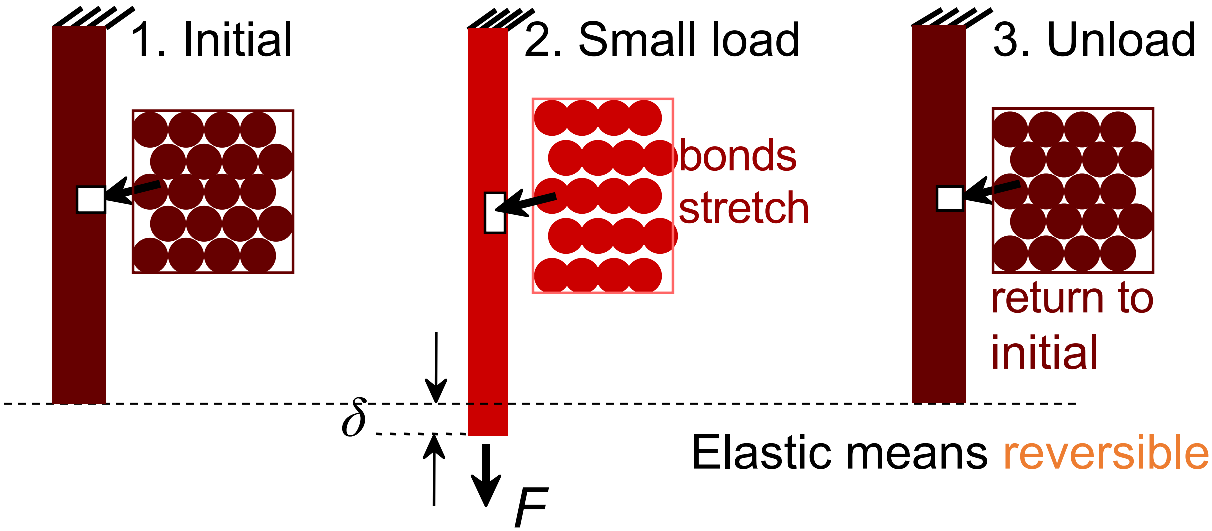


Elasticity (탄성): thought experiment

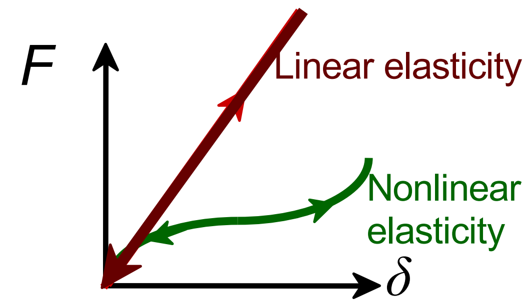
Let’s conduct a thought experiment as described below

1. Suppose you are pulling down a metallic specimen whose upper end is ‘fixed’

3. The pulling force decreases and eventually you let it go.



Elastic means reversible!



2. You are pulling the specimen harder and harder – meaning that the force at the lower end increase.

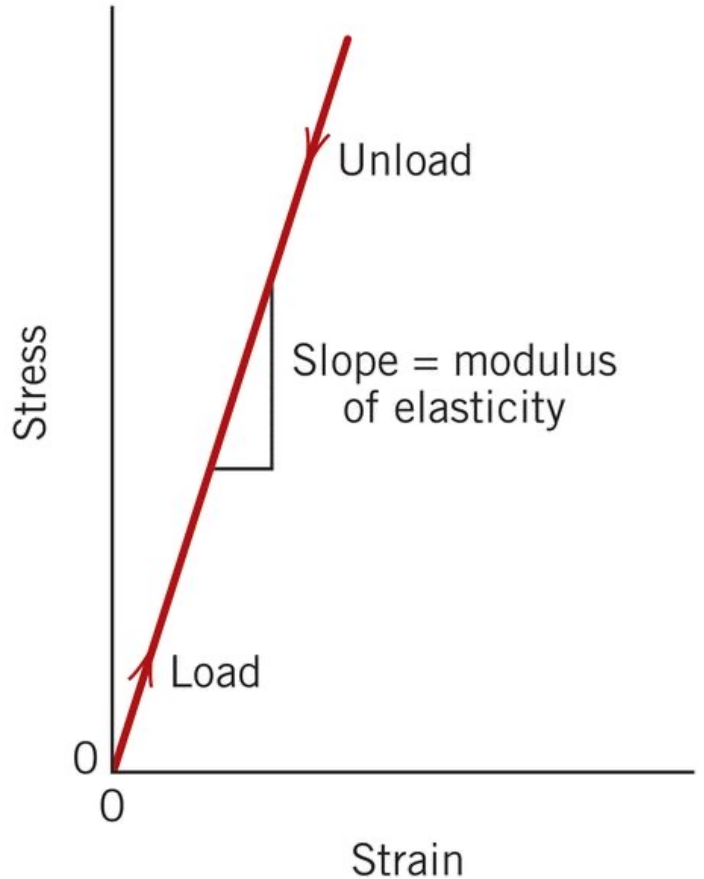
Usually metals show linear-elasticity

Non-linear elasticity is observed in polymers, rubbers

* 응력과 변형률이 (선형) 비례성이 탄성을 의미하지 않는다. 탄성은 작용응력이 제거된 후 작용 전으로의 모습으로 ‘복원’ 하느냐가 기준이다.

탄성 계수, 전단 계수

- **<u>탄성 계수</u>**𝔼는 normal component (즉 인장 혹은 압축 응력과 변형률) 사이에서의 비례상수
- 전단 계수는 shear component (즉 전단 응력과 전단 변형률)간의 비례상수
- Hooke's law에는 선형관계가 가정되어 있다. 거의 대부분의 금속은 탄성 구간에서 선형성을 나타낸다: 즉 응력과 전단 변형률이 서로 선형 비례한다.



탄성 변형률과 원자 결합력

- 앞서 살펴본 것과 같이 탄성 구간에서는 원자간의 거리가 외부의 힘에 의해 변하며, 원자간의 결합 거리가 늘어난 상태로 볼 수 있다.

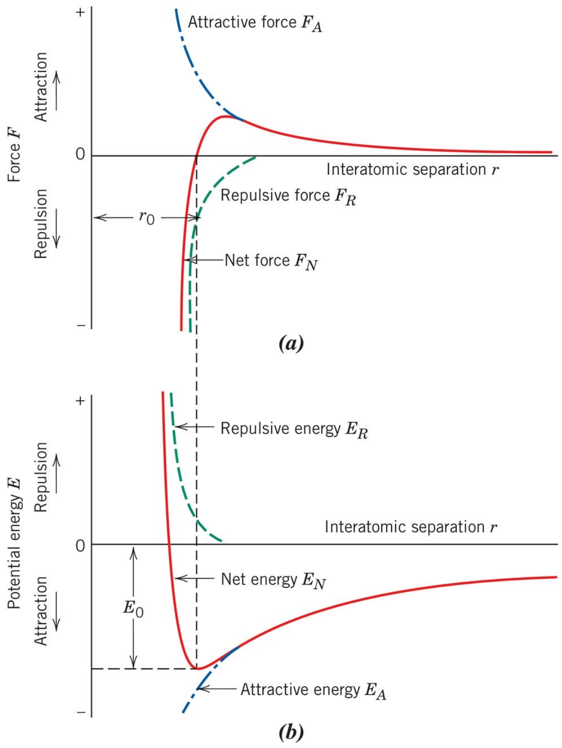
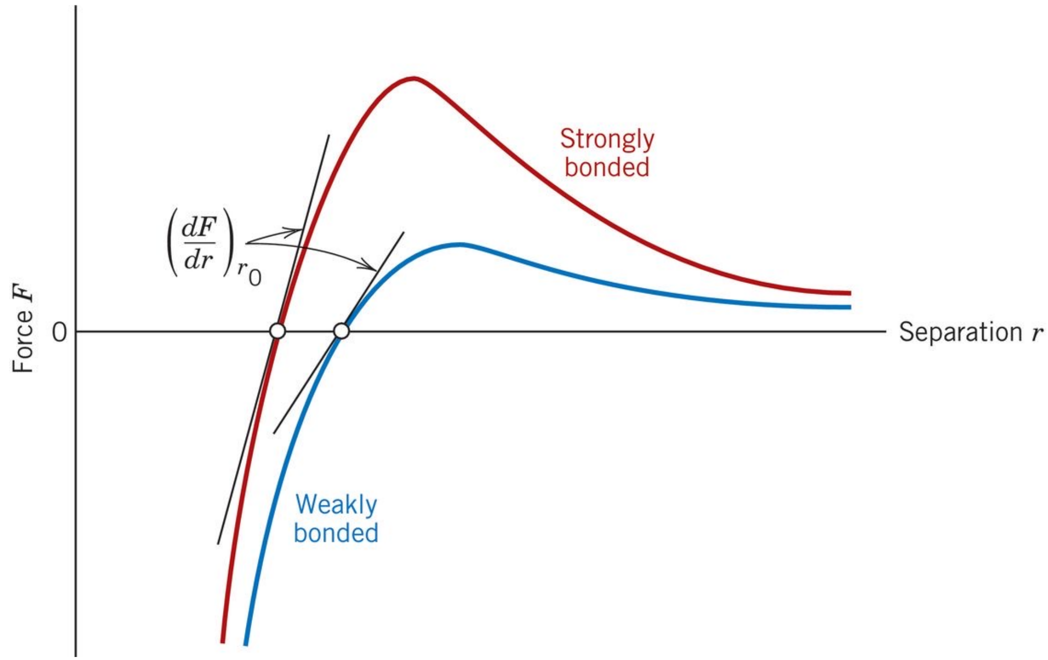


Fig. 2.10a



- 평형상태에서 원자 거리를 변화시킬 때 필요한 '힘'은 위의 그래프에서의 기울기 (이 기울기가 탄성 계수와 관계)
- 재료마다 해당 점에서의 기울기가 다를 수 있다 (각 재료의 특성)

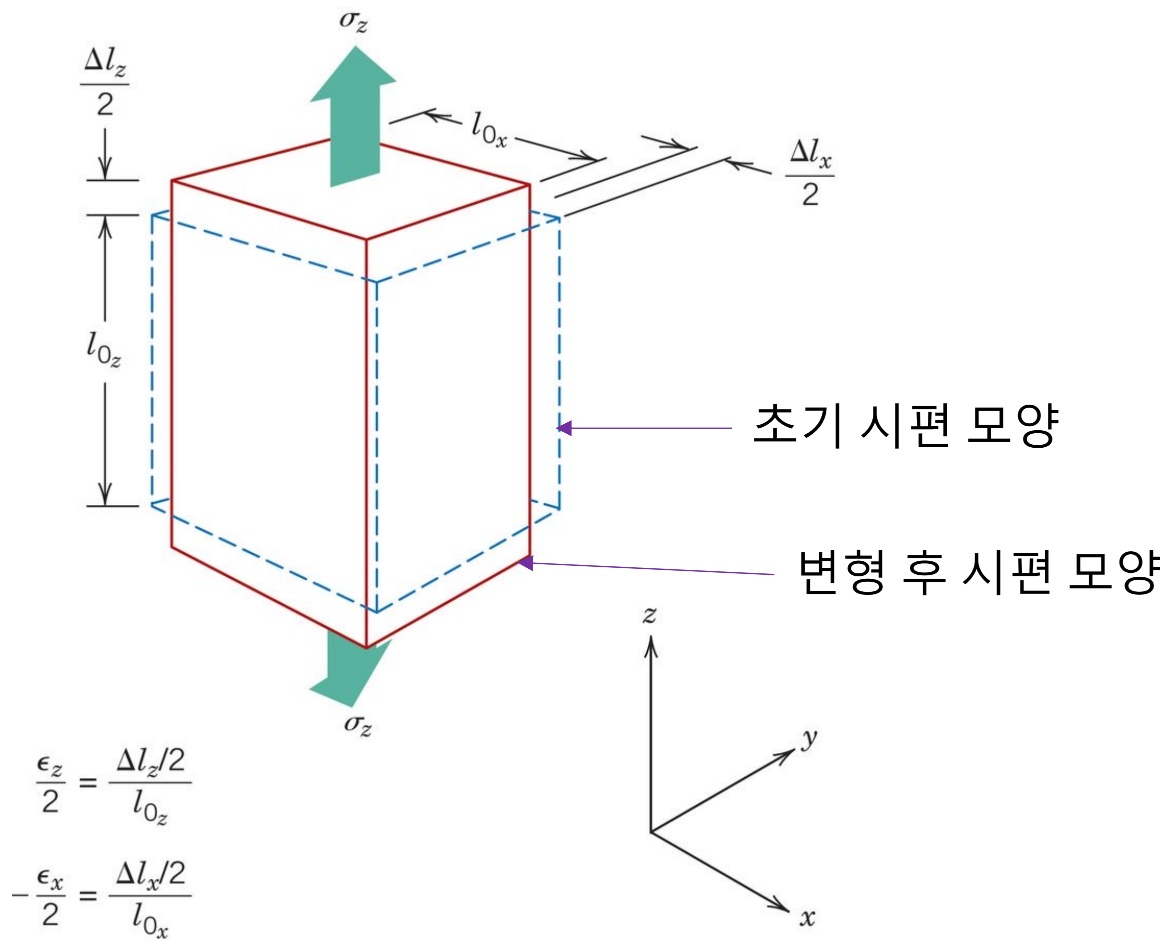
Poisson ratio (푸아송 비) (ν; nu symbol)

축 방향으로 작용하는 일축 normal stress 방향에 의해서 해당 힘과 같은 방향으로만 변형이 일어나는 것이 아니라 그것과 '수직'으로도 변형이 발생한다.

$$\nu = -\frac{\varepsilon_x}{\varepsilon_z} = -\frac{\varepsilon_y}{\varepsilon_z}$$

z축 방향 인장시 $\varepsilon_y < 0$

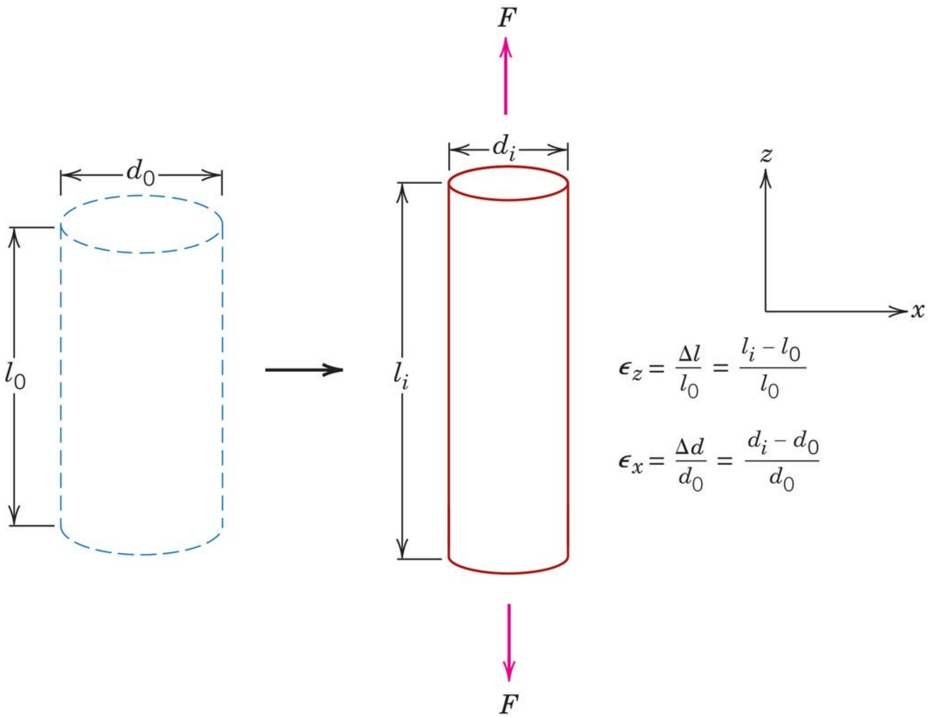
등방성(isotropy) 가진 경우



Elasticity 예제

- 10mm의 지름을 가진 황동 막대에 장축 방향으로 인장 응력 작용시켜 지름을 $2.5\times10^{-3}mm$ 로 수축시키는데 필요한 하중(힘)을 구하라. 변형은 완전 탄성으로 가정.

Metal Alloy	Modulus of Elasticity GPa	Shear Modulus GPa	Poisson's Ratio
Aluminum	69	25	0.33
Brass	97	37	0.34
Copper	110	46	0.34
Magnesium	45	17	0.29
Nickel	207	76	0.31
Steel	207	83	0.30
Titanium	107	45	0.34
Tungsten	407	160	0.28



$$F = \sigma A_0 = \sigma\left(\frac{d_0}{2}\right)^2 \pi \quad (1)$$

$$\nu = -\frac{\varepsilon_x}{\varepsilon_z} = 0.34 \quad (2)$$

$$\varepsilon_x = \frac{\Delta d}{d_0} = \frac{-2.5\times10^{-3}}{10} \quad (3)$$

With (2) and (3), you'll get ε_z

Knowing $\mathbb{E}$ andε_z you can get the stress

$$\sigma = \mathbb{E}\varepsilon_z$$

Using (1), you can obtain the force.

Constitutive description on elasticity

Elastic constitutive law (in tensor notation):

$$\mathbb{E}\colon \boldsymbol{\varepsilon} = \boldsymbol{\sigma}$$

Elastic constitutive law (in indicial notation):

$\mathbb{E}_{ijkl}\varepsilon_{kl} = \sigma_{ij}$ (two free index: i, j while, two non-free indices: k, l)

Apply it to FORTRAN, (or Python or Excel)

-Exercise 1. [3x1] = [3x3] [3x1]

-Exercise 2. [nx1] = [nxn] [nx1]

- (hint): use $[A]_i = [B]_{ij}[C]_j$

Kronecker delta may appear in formula

$$a = \varepsilon_{kl}\delta_{kl} = \sum_k \sum_l \varepsilon_{kl}\delta_{kl} = \sum_l \varepsilon_{1l}\delta_{1l} + \sum_l \varepsilon_{2l}\delta_{2l} + \sum_l \varepsilon_{3l}\delta_{3l}$$

$$= \varepsilon_{11}\delta_{11} + \varepsilon_{22}\delta_{22} + \varepsilon_{33}\delta_{33} = \sum_k \varepsilon_{kk} = \sum_i \varepsilon_{ii} = \sum_l \varepsilon_{ll}$$

$$= \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

Linear isotropic elasticity

Elastic constitutive law (Hooke's law):

$$\mathbb{E}_{ijkl}\varepsilon_{kl} = \sigma_{ij} \qquad (linear\ elasticity)$$

$$\mathbb{E}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \qquad (isotropic\ elasticity;\ two\ constants\ \lambda, \mu)$$

Replacing $\mathbb{E}_{ijkl}$ to the Hooke's law

$$\begin{aligned}\sigma_{ij} &= \mathbb{E}_{ijkl}\varepsilon_{kl} = \lambda\delta_{ij}\delta_{kl}\varepsilon_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\varepsilon_{kl} \\ &= \lambda\delta_{ij}\varepsilon_{kk} + \mu\left(\delta_{ik}\delta_{jl}\varepsilon_{kl} + \delta_{il}\delta_{jk}\varepsilon_{kl}\right) = \lambda\delta_{ij}\varepsilon_{kk} + \mu\left(\delta_{ik}\varepsilon_{kj} + \delta_{il}\varepsilon_{jl}\right) \\ &= \lambda\delta_{ij}\varepsilon_{kk} + \mu\left(\varepsilon_{ij} + \varepsilon_{ji}\right) = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}\end{aligned}$$

Demonstration with Excel

Examples

- In order for a material (with $\lambda = 115.384\ \text{GPa}, \mu = 76.923\ \text{GPa}$) to exhibit below elastic strain, what stress should be given?

$$\varepsilon = \begin{bmatrix} 0.002 & 0 & 0 \\ 0 & -0.0006 & 0 \\ 0 & 0 & -0.0006 \end{bmatrix}$$

- Hint: use "$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu\varepsilon_{ij}$"

```
200.000    0.000    0.000
  0.000    0.000    0.000
  0.000    0.000    0.000
```

Linear isotropic elasticity (Young, Poisson)

Elastic constitutive law (Hooke's law):

$$\varepsilon_{ij} = \frac{1}{E}\left[\sigma_{ij} - \nu\left(\sigma_{kk}\delta_{ij} - \sigma_{ij}\right)\right]$$

- If you apply below stress to a material (with $E = 200\ \text{GPa}, \nu = 0.3$), in what strain tensor will the material exhibit?

$$\sigma = \begin{bmatrix} 200 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \text{the unit of stress is MPa}$$

```
0.00100  0.00000  0.00000
0.00000 -0.00030  0.00000
0.00000  0.00000 -0.00030
```

Notice that a material with $E = 200\ \text{GPa}, \nu = 0.3$ behaves equivalently with a material with $\lambda = 115.384\ \text{GPa}, \mu = 76.923\ \text{GPa}$

Demonstration with Excel

Symmetries; why only two parameters?

- $\sigma_{ij} = \sigma_{ji}$ gives $\mathbb{E}_{ijkl} = \mathbb{E}_{jikl}$ thus, the required number of elastic constants reduces from 3x3x3x3 to 6x3x3.

- Similarly, $\varepsilon_{ij} = \varepsilon_{ji}$ gives $\mathbb{E}_{ijkl} = \mathbb{E}_{ijlk}$ so that we have the required of number of constants 6x6=36

The required number of constants can be further reduced. Consider the elastic energy:

$$\phi = \int \sigma_{ij} d\varepsilon_{ij}$$

$$\sigma_{ij} = \frac{\partial \phi}{\partial \varepsilon_{ij}} = \mathbb{E}_{ijkl}\varepsilon_{kl}$$

If we apply partial derivative once again, we have

$\frac{\partial^2 \phi}{\partial \varepsilon_{mn} \partial \varepsilon_{ij}} = \frac{\partial}{\partial \varepsilon_{mn}}\left(\mathbb{E}_{ijkl}\varepsilon_{kl}\right)$ since $\mathbb{E}$ is 'constant', we have

$$\frac{\partial^2 \phi}{\partial \varepsilon_{mn} \partial \varepsilon_{ij}} = \mathbb{E}_{ijkl}\left(\frac{\partial \varepsilon_{kl}}{\partial \varepsilon_{mn}}\right) = \mathbb{E}_{ijkl}\delta_{km}\delta_{ln} = \mathbb{E}_{ijmn}$$

Symmetries; why only two parameters?

- $\phi = \int \sigma_{ij} d\varepsilon_{ij}$
- $\sigma_{ij} = \frac{\partial \phi}{\partial \varepsilon_{ij}} = \mathbb{E}_{ijkl}\varepsilon_{kl}$
- If we apply partial derivative once again, we have
- $\frac{\partial^2 \phi}{\partial \varepsilon_{mn}\partial \varepsilon_{ij}} = \frac{\partial}{\partial \varepsilon_{mn}}\left(\mathbb{E}_{ijkl}\varepsilon_{kl}\right)$ since $\mathbb{E}$ is 'constant', we have
- $\frac{\partial^2 \phi}{\partial \varepsilon_{mn}\partial \varepsilon_{ij}} = \mathbb{E}_{ijkl}\left(\frac{\partial \varepsilon_{kl}}{\partial \varepsilon_{mn}}\right) = \mathbb{E}_{ijkl}\delta_{km}\delta_{ln} = \mathbb{E}_{ijmn}$

- We could do the 2nd order derivative in a different way (say, instead of $\frac{\partial^2 \phi}{\partial \varepsilon_{mn}\partial \varepsilon_{ij}}$ we could have done $\frac{\partial^2 \phi}{\partial \varepsilon_{ij}\partial \varepsilon_{mn}} = \frac{\partial}{\partial \varepsilon_{ij}}\left(\frac{\partial \phi}{\partial \varepsilon_{mn}}\right) = \frac{\partial}{\partial \varepsilon_{ij}}\left(\frac{\partial \phi}{\partial \varepsilon_{mn}}\right) = \mathbb{E}_{mnij}$
- The two cases (regardless of the order of derivative) should give equivalent result so that
- $\mathbb{E}_{ijmn} = \mathbb{E}_{mnij}$

- This summarizes our finding on the symmetries in elastic tensor:

Reduction to Voigt notation

- $\sigma_{21} = \mathbb{E}_{2111}\varepsilon_{11} + \mathbb{E}_{2112}\varepsilon_{12} + \mathbb{E}_{2113}\varepsilon_{13} + \mathbb{E}_{2121}\varepsilon_{21} + \mathbb{E}_{2122}\varepsilon_{22} + \mathbb{E}_{2123}\varepsilon_{23} + \mathbb{E}_{2131}\varepsilon_{31} + \mathbb{E}_{2132}\varepsilon_{32} + \mathbb{E}_{2123}\varepsilon_{33}$

- $$\sigma_{21} = \begin{bmatrix} \mathbb{E}_{2111} \\ \mathbb{E}_{2112} \\ \mathbb{E}_{2113} \\ \mathbb{E}_{2121} \\ \mathbb{E}_{2122} \\ \mathbb{E}_{2123} \\ \mathbb{E}_{2131} \\ \mathbb{E}_{2132} \\ \mathbb{E}_{2133} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{21} \\ \varepsilon_{22} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{32} \\ \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \mathbb{E}_{2111} \\ 2\mathbb{E}_{2112} \\ 2\mathbb{E}_{2113} \\ - \\ \mathbb{E}_{2122} \\ 2\mathbb{E}_{2123} \\ - \\ - \\ \mathbb{E}_{2133} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ - \\ \varepsilon_{22} \\ \varepsilon_{23} \\ - \\ - \\ \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \mathbb{E}_{2111} \\ \mathbb{E}_{2122} \\ \mathbb{E}_{2133} \\ 2\mathbb{E}_{2123} \\ 2\mathbb{E}_{2113} \\ 2\mathbb{E}_{2112} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{bmatrix}$$

- $$or \quad = \begin{bmatrix} \mathbb{E}_{2111} \\ \mathbb{E}_{2122} \\ \mathbb{E}_{2133} \\ \mathbb{E}_{2123} \\ \mathbb{E}_{2113} \\ \mathbb{E}_{2112} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix}$$ with $\gamma_{12} = 2\varepsilon_{12}$ and so forth

$$\sigma_{21} = \begin{bmatrix} \mathbb{E}_{21,1} \\ \mathbb{E}_{21,2} \\ \mathbb{E}_{21,3} \\ \mathbb{E}_{21,4} \\ \mathbb{E}_{21,5} \\ \mathbb{E}_{21,6} \end{bmatrix} \begin{bmatrix} \varepsilon_{1} \\ \varepsilon_{2} \\ \varepsilon_{3} \\ \gamma_{4} \\ \gamma_{5} \\ \gamma_{6} \end{bmatrix}$$

with $(1,1) \rightarrow (1), (2,2) \rightarrow (2), (3,3) \rightarrow (3)$
$(2,3) \rightarrow (4), (1,3) \rightarrow (5), (1,2) \rightarrow (6)$

Reduction to Voigt notation

$$\sigma_{21} = \begin{bmatrix} \mathbb{E}_{21,1} \\ \mathbb{E}_{21,2} \\ \mathbb{E}_{21,3} \\ \mathbb{E}_{21,4} \\ \mathbb{E}_{21,5} \\ \mathbb{E}_{21,6} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{bmatrix}$$

with $(1,1) \rightarrow (1), (2,2) \rightarrow (2), (3,3) \rightarrow (3)$
$(2,3) \rightarrow (4), (1,3) \rightarrow (5), (1,2) \rightarrow (6)$

$$\sigma_{21} = \begin{bmatrix} \mathbb{E}_{21,1} \\ \mathbb{E}_{21,2} \\ \mathbb{E}_{21,3} \\ \mathbb{E}_{21,4} \\ \mathbb{E}_{21,5} \\ \mathbb{E}_{21,6} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}$$

with$(1,2) \rightarrow (2,1) \rightarrow (6)$

$$\sigma_{6} = \begin{bmatrix} \mathbb{E}_{6,1} \\ \mathbb{E}_{6,2} \\ \mathbb{E}_{6,3} \\ \mathbb{E}_{6,4} \\ \mathbb{E}_{6,5} \\ \mathbb{E}_{6,6} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}$$

with$(1,2) \rightarrow (2,1) \rightarrow (6)$

Reduction to Voigt notation

$$\sigma_{ij} = \mathbb{E}_{ijkl}\varepsilon_{kl}$$

$$\sigma_i = \mathbb{E}_{ij}\varepsilon_j$$

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix}\mathbb{E}_{1111} & \mathbb{E}_{1122} & \mathbb{E}_{1133} & \mathbb{E}_{1123} & \mathbb{E}_{1113} & \mathbb{E}_{1112}\\ \mathbb{E}_{2211} & \mathbb{E}_{2222} & \mathbb{E}_{2233} & \mathbb{E}_{2223} & \mathbb{E}_{2213} & \mathbb{E}_{2212}\\ \mathbb{E}_{3311} & \mathbb{E}_{3322} & \mathbb{E}_{3333} & \mathbb{E}_{3323} & \mathbb{E}_{3313} & \mathbb{E}_{3312}\\ \mathbb{E}_{2311} & \mathbb{E}_{2322} & \mathbb{E}_{2333} & \mathbb{E}_{2323} & \mathbb{E}_{2313} & \mathbb{E}_{2312}\\ \mathbb{E}_{1311} & \mathbb{E}_{1322} & \mathbb{E}_{1333} & \mathbb{E}_{1323} & \mathbb{E}_{1313} & \mathbb{E}_{1312}\\ \mathbb{E}_{1211} & \mathbb{E}_{1222} & \mathbb{E}_{1233} & \mathbb{E}_{1223} & \mathbb{E}_{1213} & \mathbb{E}_{1212}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{12}\end{bmatrix}$$

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{bmatrix} = \begin{bmatrix}\mathbb{E}_{11} & \mathbb{E}_{12} & \mathbb{E}_{13} & \mathbb{E}_{14} & \mathbb{E}_{15} & \mathbb{E}_{16}\\ \mathbb{E}_{21} & \mathbb{E}_{22} & \mathbb{E}_{23} & \mathbb{E}_{24} & \mathbb{E}_{25} & \mathbb{E}_{26}\\ \mathbb{E}_{31} & \mathbb{E}_{32} & \mathbb{E}_{33} & \mathbb{E}_{34} & \mathbb{E}_{35} & \mathbb{E}_{36}\\ \mathbb{E}_{41} & \mathbb{E}_{42} & \mathbb{E}_{43} & \mathbb{E}_{44} & \mathbb{E}_{45} & \mathbb{E}_{46}\\ \mathbb{E}_{51} & \mathbb{E}_{52} & \mathbb{E}_{53} & \mathbb{E}_{54} & \mathbb{E}_{55} & \mathbb{E}_{56}\\ \mathbb{E}_{61} & \mathbb{E}_{62} & \mathbb{E}_{63} & \mathbb{E}_{64} & \mathbb{E}_{65} & \mathbb{E}_{66}\end{bmatrix}\begin{bmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_3\\ \epsilon_4\\ \epsilon_5\\ \epsilon_6\end{bmatrix}$$

How many constants are required?

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix}\mathbb{E}_{1111} & \mathbb{E}_{1122} & \mathbb{E}_{1133} & \mathbb{E}_{1123} & \mathbb{E}_{1113} & \mathbb{E}_{1112}\\ \mathbb{E}_{2211} & \mathbb{E}_{2222} & \mathbb{E}_{2233} & \mathbb{E}_{2223} & \mathbb{E}_{2213} & \mathbb{E}_{2212}\\ \mathbb{E}_{3311} & \mathbb{E}_{3322} & \mathbb{E}_{3333} & \mathbb{E}_{3323} & \mathbb{E}_{3313} & \mathbb{E}_{3312}\\ \mathbb{E}_{2311} & \mathbb{E}_{2322} & \mathbb{E}_{2333} & \mathbb{E}_{2323} & \mathbb{E}_{2313} & \mathbb{E}_{2312}\\ \mathbb{E}_{1311} & \mathbb{E}_{1322} & \mathbb{E}_{1333} & \mathbb{E}_{1323} & \mathbb{E}_{1313} & \mathbb{E}_{1312}\\ \mathbb{E}_{1211} & \mathbb{E}_{1222} & \mathbb{E}_{1233} & \mathbb{E}_{1223} & \mathbb{E}_{1213} & \mathbb{E}_{1212}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{12}\end{bmatrix}$$

How many constants do we need?

If the coordinate system happens to give strain and stress all principal values:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{bmatrix} = \begin{bmatrix} \mathbb{E}_{1111} & \mathbb{E}_{1122} & \mathbb{E}_{1133} \\ & \mathbb{E}_{2222} & \mathbb{E}_{2223} \\ & & \mathbb{E}_{3333} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \end{bmatrix}$$

example

- Fe(1-0.025)-Al(0.025) alloy의 탄성 계수는 다음과 같이 주어진다.

- $\mathbb{E}_{11}$ = 270.71, $\mathbb{E}_{12}$ = 128.03, $\mathbb{E}_{44}$ = 108.77
- Fe-Al alloy는 Body-centered cubic 결정 구조를 가지고, 결정 대칭성에 의해 다음과 같은 탄성 거동을 한다.

$$\begin{bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\sigma_4\\\sigma_5\\\sigma_6\end{bmatrix}=\begin{bmatrix}\mathbb{E}_{11}&\mathbb{E}_{12}&\mathbb{E}_{13}&0&0&0\\\mathbb{E}_{21}&\mathbb{E}_{22}&\mathbb{E}_{23}&0&0&0\\\mathbb{E}_{31}&\mathbb{E}_{32}&\mathbb{E}_{33}&0&0&0\\0&0&0&\mathbb{E}_{44}&0&0\\0&0&0&0&\mathbb{E}_{55}&0\\0&0&0&0&0&\mathbb{E}_{66}\end{bmatrix}\begin{bmatrix}\epsilon_1\\\epsilon_2\\\epsilon_3\\\epsilon_4\\\epsilon_5\\\epsilon_6\end{bmatrix}$$

- 뿐만 아니라, cubic 결정구조의 대칭성으로 인해 $\mathbb{E}_{11}=\mathbb{E}_{22}=\mathbb{E}_{33}, \mathbb{E}_{44}=\mathbb{E}_{55}=\mathbb{E}_{66}, \mathbb{E}_{12}=\mathbb{E}_{13}=\mathbb{E}_{23}$

Example

- Fe(1-0.025)-Al(0.025) alloy의 단결정에 다음과 같은 탄성 변형률이 나타나기 위해 필요한 응력 상태는?

$$\begin{bmatrix} 0.0001 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Voigt notation

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112}\\ & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212}\\ & & C_{3333} & C_{3323} & C_{3313} & C_{3312}\\ & & & C_{2323} & C_{2313} & C_{2312}\\ & symm & & & C_{1313} & C_{1312}\\ & & & & & C_{1212}\end{bmatrix} \begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{33}\\ 2\epsilon_{23}\\ 2\epsilon_{13}\\ 2\epsilon_{12}\end{bmatrix}$$

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ & & C_{33} & C_{34} & C_{35} & C_{36}\\ & & & C_{44} & C_{45} & C_{46}\\ & symm & & & C_{55} & C_{56}\\ & & & & & C_{66}\end{bmatrix} \begin{bmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_3\\ \epsilon_4\\ \epsilon_5\\ \epsilon_6\end{bmatrix}$$

Cartesian <-> Voigt (cheat sheet)

```fortran
c      stress, stiffness
       SUBROUTINE VOIGT(T1,T2,C2,C4,IOPT)
       IMPLICIT NONE
       REAL*8 T1(6),T2(3,3),C2(6,6),C4(3,3,3,3)
       INTEGER IJV(6,2),I,J,IOPT,I1,I2,J1,J2,N,M
       DATA ((IJV(N,M),M=1,2),N=1,6)/1,1,2,2,3,3,1,2,1,3,2,3/

       IF(IOPT.EQ.1) THEN
       DO I=1,6
          I1=IJV(I,1)
          I2=IJV(I,2)
          T2(I1,I2)=T1(I)
          T2(I2,I1)=T1(I)
       ENDDO
       ENDIF
C
       IF(IOPT.EQ.2) THEN
          DO I=1,6
             I1=IJV(I,1)
             I2=IJV(I,2)
             T1(I)=T2(I1,I2)
          ENDDO
          ENDIF
C
       IF (IOPT.EQ.3) THEN
          DO I=1,6
             I1=IJV(I,1)
             I2=IJV(I,2)
             DO J=1,6
                J1=IJV(J,1)
                J2=IJV(J,2)
                C4(I1,I2,J1,J2)=C2(I,J)
                C4(I2,I1,J1,J2)=C2(I,J)
                C4(I1,I2,J2,J1)=C2(I,J)
                C4(I2,I1,J2,J1)=C2(I,J)
             ENDDO
          ENDDO
       ENDIF
C
       IF(IOPT.EQ.4) THEN
          DO  I=1,6
             I1=IJV(I,1)
             I2=IJV(I,2)
             DO  J=1,6
                J1=IJV(J,1)
                J2=IJV(J,2)
                C2(I,J)=C4(I1,I2,J1,J2)
             ENDDO
          ENDDO
       ENDIF
C
       RETURN
       END
```

Convert from Cartesian to Voigt

- $\mathbb{E}_{11}^{(\text{voigt})} = \mathbb{E}_{1111}^{(\text{cartesian})}, \mathbb{E}_{23}^{(\text{voigt})} = \mathbb{E}_{2233}^{(\text{cartesian})}, \mathbb{E}_{41}^{(\text{voigt})} = \mathbb{E}_{2311}^{(\text{cartesian})}$

- Material anisotropy
- Symmetry can be represented by an orthogonal second order tensor,
- $\boldsymbol{Q} = Q_{ij}\mathbf{e}_{\mathrm{i}} \otimes \mathbf{e}_{\mathrm{j}}$, such that $\boldsymbol{Q}^{-1} = \boldsymbol{Q}^{T}$
- The invariance of the stiffness tensor under these transformations (due to symmetry) is:
- $\mathbb{E}^{(new)} = \boldsymbol{Q} \cdot \boldsymbol{Q} \cdot \mathbb{E}^{(old)} \cdot \boldsymbol{Q}^{T} \cdot \boldsymbol{Q}^{T}$ due to symmetry the resulting tensor should be equivalent with the original one: $\mathbb{E}^{(new)} \equiv \mathbb{E}^{(old)}$

http://web.mit.edu/16.20/homepage/3_Constitutive/Constitutive_files/module_3_with_solutions.pdf

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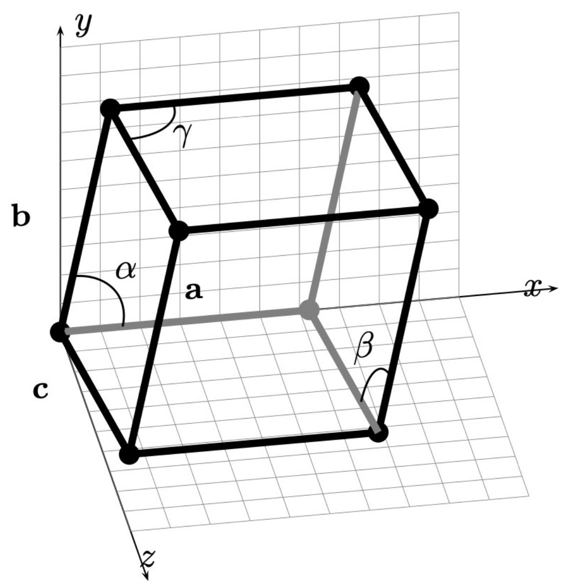
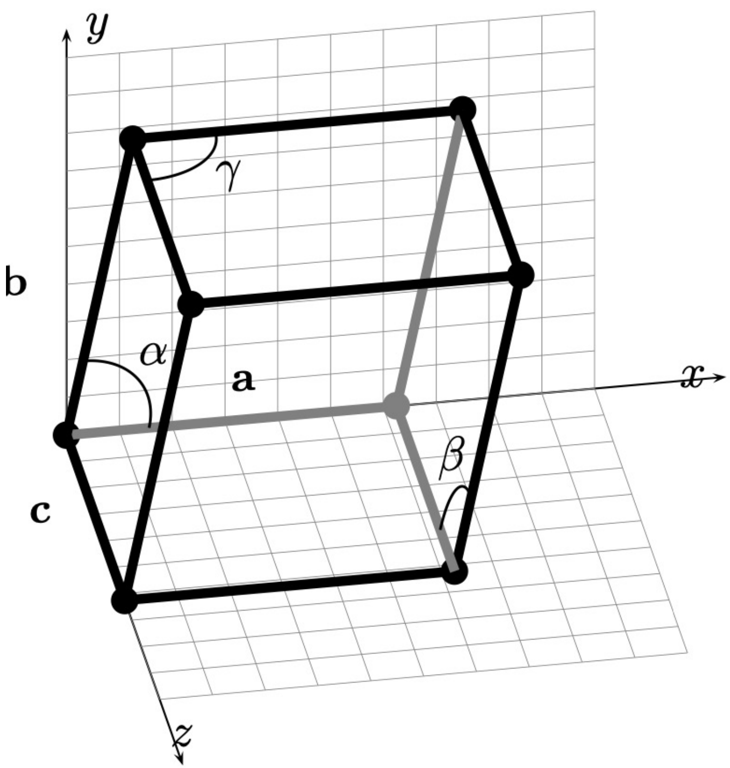
Triclinic (no symmetry)



Triclinic: no symmetry planes, fully anisotropic.
$\alpha, \beta, \gamma < 90$
Number of independent coefficients: 21
Symmetry transformation: None

$$C = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ & & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ & & & C_{2323} & C_{2313} & C_{2312} \\ & symm & & & C_{1313} & C_{1312} \\ & & & & & C_{1212} \end{bmatrix}$$

monoclinic (one symmetry plane)



Monoclinic: one symmetry plane (xy). $a \neq b \neq c,\ \beta = \gamma = 90, \alpha < 90$
Number of independent coefficients: 13
Symmetry transformation: reflection about z-axis

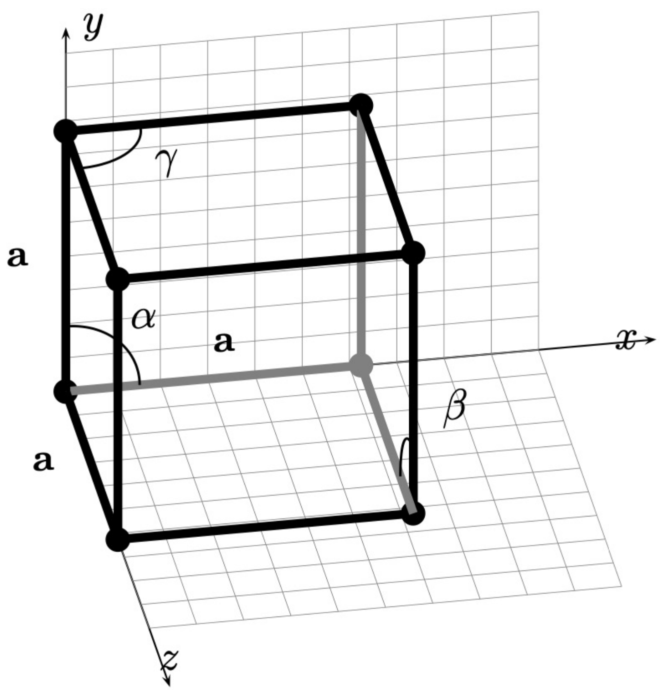
$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & C_{1112} \\ & C_{2222} & C_{2233} & 0 & 0 & C_{2212} \\ & & C_{3333} & 0 & 0 & C_{3312} \\ & & & C_{2323} & C_{2313} & 0 \\ & symm & & & C_{1313} & 0 \\ & & & & & C_{1212} \end{bmatrix}$$

monoclinic (one symmetry plane)

- For the case of Monoclinic:
- $\boldsymbol{Q} = Q_{ij}\mathbf{e}_\mathrm{i} \otimes \mathbf{e}_\mathrm{j} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
- Let’s take a look at the invariance due to symmetry
- $\mathbb{E}^{(new)} = \boldsymbol{Q} \cdot \boldsymbol{Q} \cdot \mathbb{E}^{(old)} \cdot \boldsymbol{Q}^T \cdot \boldsymbol{Q}^T$ due to symmetry the resulting tensor should be equivalent with the original one: $\mathbb{E}^{(new)} \equiv \mathbb{E}^{(old)}$
- In its matrix form:
  - $\mathbb{E}_{ijkl}^{(new)} = Q_{im} Q_{jn} Q_{ko} Q_{lp} \mathbb{E}_{mnop}^{(old)}$
  - Ex: $\mathbb{E}_{11}^{(voigt)} = \mathbb{E}_{1111} = Q_{1m} Q_{1n} Q_{1o} Q_{1p} \mathbb{E}_{mnop}^{(old)}$
    - If you look at the matrix form of symmetry operator Q in the above, only diagonal components are non-zero. Therefore, $Q_{ij} = 0$ if $i \neq j$.
    - $\mathbb{E}_{11}^{(voigt)} = \mathbb{E}_{1111} = Q_{11} Q_{11} Q_{11} Q_{11} \mathbb{E}_{1111}^{(old)} = \mathbb{E}_{1111}^{(old)}$
    - Therefore, $\mathbb{E}_{11}^{(voigt)} = \mathbb{E}_{1111}$
  - Ex: $\mathbb{E}_{14}^{(voigt)} = \mathbb{E}_{1123} = Q_{1m} Q_{1n} Q_{2o} Q_{3p} \mathbb{E}_{mnop}^{(old)} = Q_{11} Q_{11} Q_{22} Q_{33} \mathbb{E}_{1123}^{(old)} = 1 \times 1 \times 1 \times (-1) \times \mathbb{E}_{1123}^{(old)} = -\mathbb{E}_{1123}^{(old)}$
    - Therefore, in order to satisfy $\mathbb{E}_{1123} = -\mathbb{E}_{1123}$, $\mathbb{E}_{1123}$ should be zero.

Cubic



Cubic: three mutually orthogonal planes of reflection symmetry plus 90° rotation symmetry with respect to those planes. $a = b = c, \; \alpha = \beta = \gamma = 90$
Number of independent coefficients: 3
Symmetry transformations: reflections and 90° rotations about all three orthogonal planes

$$Q = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \; Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} C_{1111} & C_{1122} & C_{1122} & 0 & 0 & 0 \\ & C_{1111} & C_{1122} & 0 & 0 & 0 \\ & & C_{1111} & 0 & 0 & 0 \\ & & & C_{1212} & 0 & 0 \\ & symm & & & C_{1212} & 0 \\ & & & & & C_{1212} \end{bmatrix}$$