Force and stress tensor

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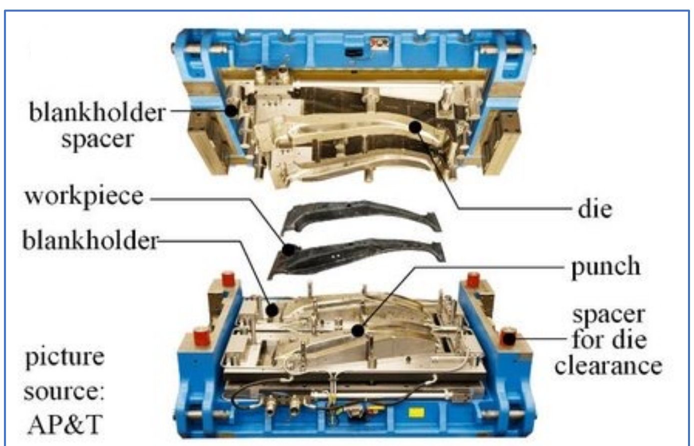
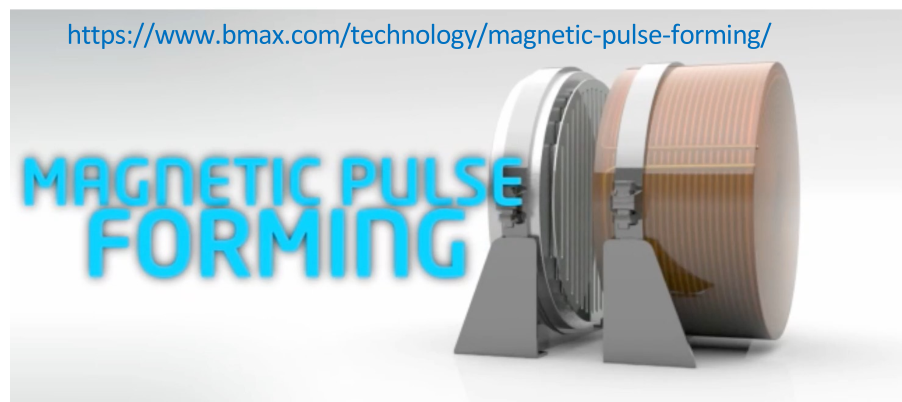
Intro

- 재료에 가해질 수 있는 힘은
  - 1) 재료 표면에 가해진 '접촉'에 의해 전달되는 힘 (surface force)
  - 2) 접촉하지 않고서 전달되는 힘 (body force)
    - 중력, 전자기력 등
- 우리가 '정역학'에서 다루는 재료의 '운동'상태 혹은 '변형'은 모두 평형 상태인 '힘'들에 의해서 나타난다. (평형상태가 아닌 힘에 의해 발생하는 현상은 다루지 않음)
- 일반적인 소성가공 공정에서 재료는 surface force에 의해서만 힘을 전달받고 변형하며, body force에 의한 소성 가공법은 매우 특별한 경우에 한정되어 있다.

Source:AP&T, (https://www.aptgroup.com)





Magnetic pulse forming
https://youtu.be/rBXXBlP9qIE

Stamping process for mass production

TESLA Stamping line: https://youtu.be/gkjn9bogLSM

Force

Force is a vector quantity, so that

$$\boldsymbol{f} = f_1\mathbf{e}_1 + f_2\mathbf{e}_2 + f_3\mathbf{e}_3$$

Force equilibrium condition:

$$\sum_{\text{source of acting force}} \boldsymbol{f}^{\text{each source}} = 0$$

$$\sum_{source} f_i = 0 \quad (free\ index\ i) \rightarrow \sum_{source} f_1 = 0, \sum_{source} f_2 = 0, \sum_{source} f_3 = 0$$

Force and moment

$\boldsymbol{m} = \boldsymbol{f} \times \boldsymbol{x}$ with $\boldsymbol{x}$ being moment arm

Moment of a force is a measure of its tendency to cause a body to 'rotation' about a specific point or axis.

$$\boldsymbol{m}_k = f_i x_j \mathbf{e}_i \times \mathbf{e}_j = f_i x_j \epsilon_{ijk} \mathbf{e}_k$$

$$= \mathbf{e}_k \sum_i \sum_j f_i x_j \epsilon_{ijk}$$

$$= \mathbf{e}_k \{ f_1 x_2 \epsilon_{12k} + f_1 x_3 \epsilon_{13k} + f_2 x_1 \epsilon_{21k} + f_2 x_3 \epsilon_{23k} + f_3 x_1 \epsilon_{31k} + f_3 x_2 \epsilon_{32k} \}$$

$$\boldsymbol{m}_1 = \mathbf{e}_1 \{ f_1 x_2 \epsilon_{121} + f_1 x_3 \epsilon_{131} + f_2 x_1 \epsilon_{211} + f_2 x_3 \epsilon_{231} + f_3 x_1 \epsilon_{311} + f_3 x_2 \epsilon_{321} \}$$

$$= \mathbf{e}_1 \{ f_2 x_3 - f_3 x_2 \}$$

$$\boldsymbol{m}_2 = \mathbf{e}_2 \{ f_1 x_2 \epsilon_{122} + f_1 x_3 \epsilon_{132} + f_2 x_1 \epsilon_{212} + f_2 x_3 \epsilon_{232} + f_3 x_1 \epsilon_{312} + f_3 x_2 \epsilon_{322} \}$$

$$= \mathbf{e}_2 \{ -f_1 x_3 + f_3 x_1 \}$$

$$\boldsymbol{m}_3 = \mathbf{e}_3 \{ f_1 x_2 - f_2 x_1 \}$$

$$m_i = 0 \text{ (moment equilibrium; } i \text{ is a free index)}$$

응력 텐서 (stress tensor)

- 재료에 전달되는 ‘힘’을 ‘세기 물리량’으로 정량화하여 나타내는데 ‘응력’이라는 물리량을 사용한다.
- 응력은 ‘텐서’ 물리량이다.
- 응력 텐서의 정의를 자세히 알아보자.
  - 힘 평형 (force equilibrium)
  - 응력 벡터 (stress vector)

힘 벡터와 응력 벡터

힘은 크기 물리량; 응력은 세기 물리량

풍선에게 동일한 힘을 손바닥으로 전달할 때와 바늘끝으로 전달할 때 풍선에게 나타날 효과가 매우 다른 것을 예상할 수 있다. 힘이 작용하는 면적의 차이에 의해서 나타나는 효과라 볼 수 있다.

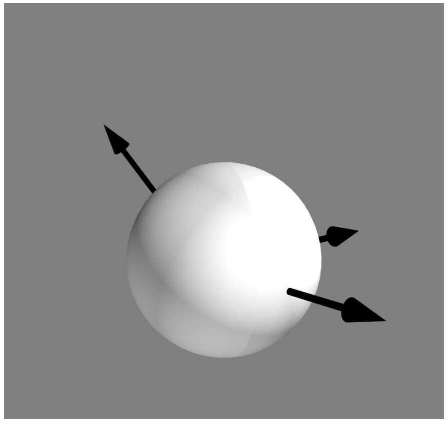
- 응력 벡터(혹은 traction이라 일컫는다)란? $\boldsymbol{t} = \lim_{\Delta S \to 0} \frac{\boldsymbol{f}}{\Delta S}$,

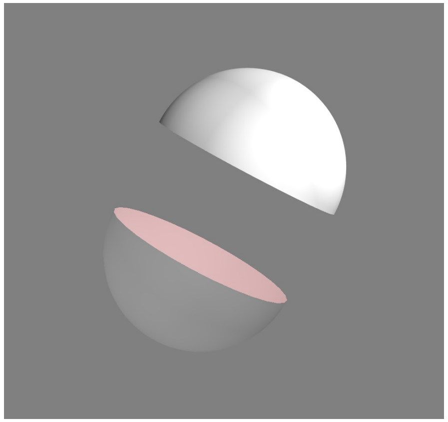
- In SI unit system, unit of $\frac{f}{\Delta S} = \frac{\left[kg\,\frac{m}{s^2}\right]}{[m^2]} = \frac{N}{m^2} = Pa$

Caution1 : 변형률은 unit이 없지만, 응력은 단위가 있다.
Caution2: 금속에 작용하는 응력은 주로 MPa가 쓰이며, 여기서 M은 Mega이며 뜻은 10^6이다.

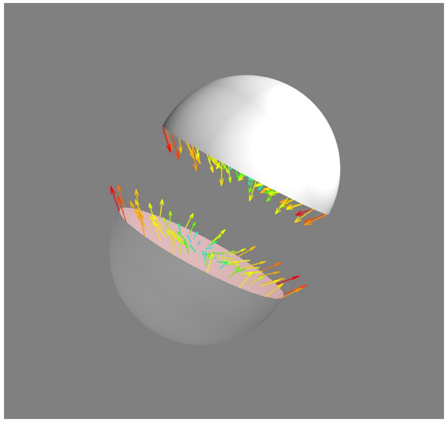
The graphical illustration to understand the definition of **stress vector (traction vector)**

외부의 surface force

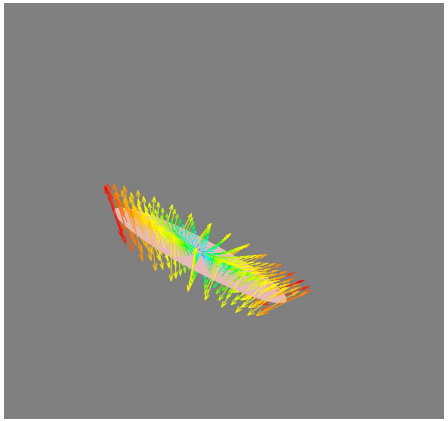




재료 내부에 force field
$\boldsymbol{f}(x, y, z)$



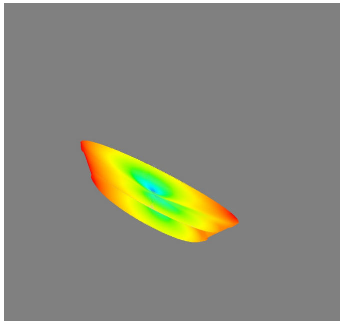
힘평형을 이룬채로
force field 발달

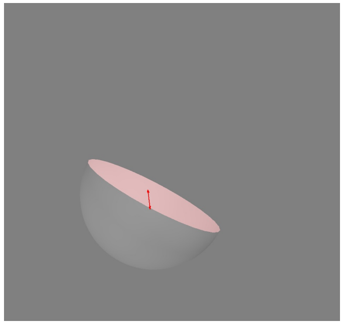


좀 더 빼곡히 나타낸
힘벡터 분포

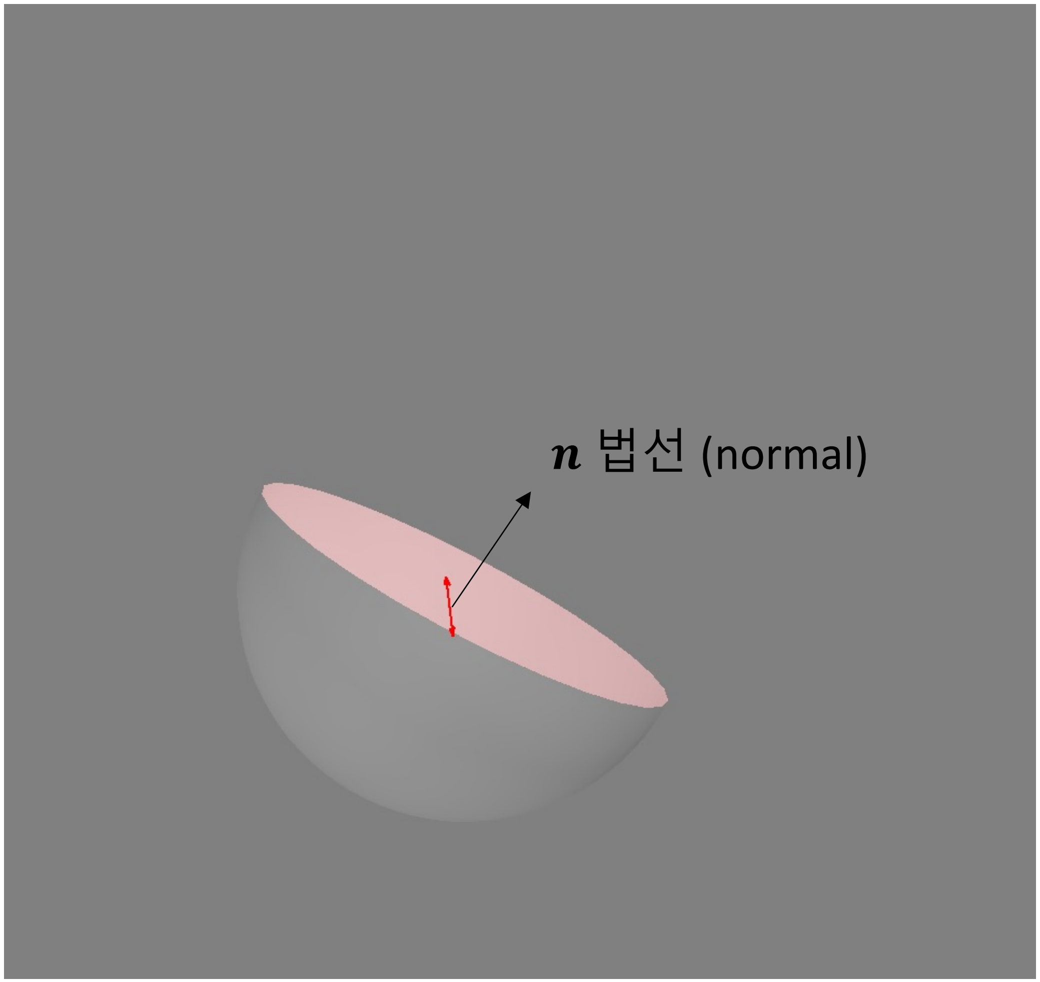
$$\boldsymbol{t} = \lim_{\Delta S \to 0} \frac{\boldsymbol{f}}{\Delta S}$$

$\boldsymbol{t}$: normal이 $\boldsymbol{n}$ 인 면에
작용하는 응력 벡터





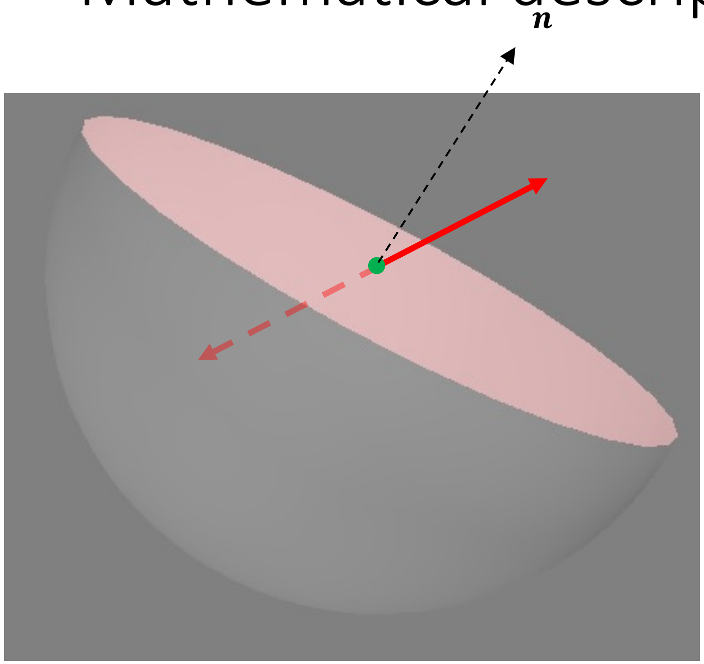
The graphical gist of it



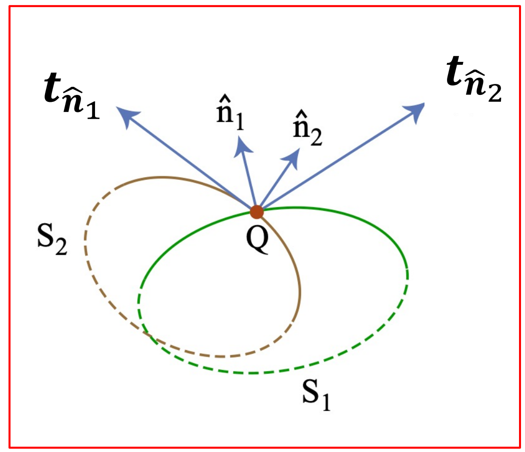
$$\boldsymbol{t} = \lim_{\Delta S \to 0} \frac{\boldsymbol{f}}{\Delta S}$$

$$\boldsymbol{t} = \frac{d\boldsymbol{f}}{dS} \qquad t_i = \frac{df_i}{dS}$$

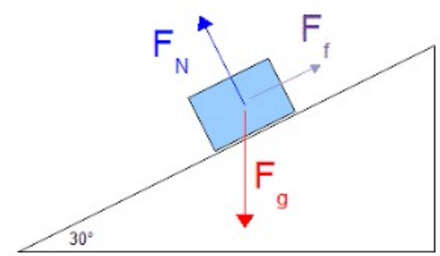
Mathematical description (both algebraic and geometrical)



$$\boldsymbol{t} = \lim_{\Delta S \to 0} \frac{\boldsymbol{f}}{\Delta S} \qquad \boldsymbol{t} = \frac{d\boldsymbol{f}}{dS} \qquad t_i = \frac{df_i}{dS}$$



물질내 고정된 한 점에서의 traction은 그 점을 통과하는 면의 방향에 따라 달라진다. 즉 $\boldsymbol{t}$는 $\boldsymbol{n}$에 따라 달라진다.



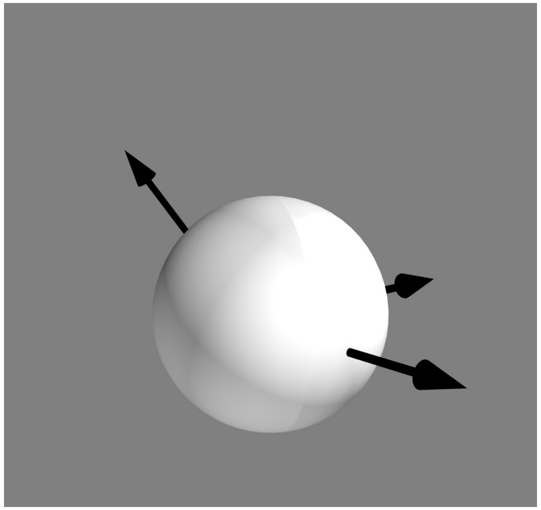
$$\boldsymbol{t} = \boldsymbol{\sigma} \cdot \boldsymbol{n} \qquad t_i = \sigma_{ij} n_j$$

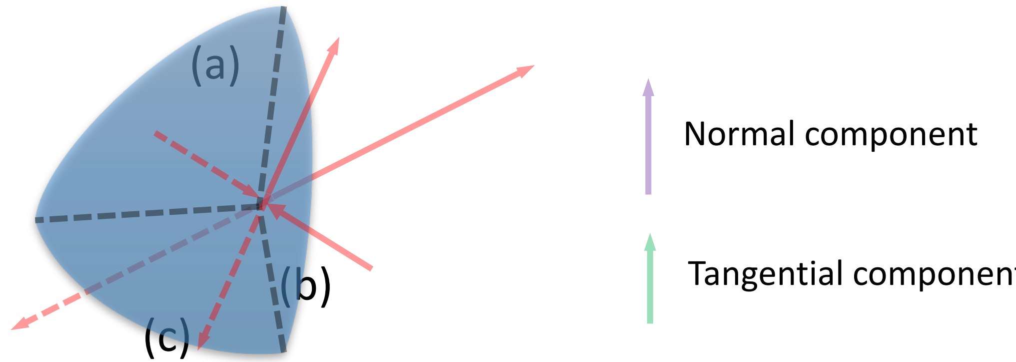
- ● The infinitesimal surface that approaches to zero (i.e., $\Delta S \to 0$)
- (Coupled) Force vectors acting on that infinitesimal surface
- Plane normal; It is a unit vector such that $|\boldsymbol{n}| = 1$.

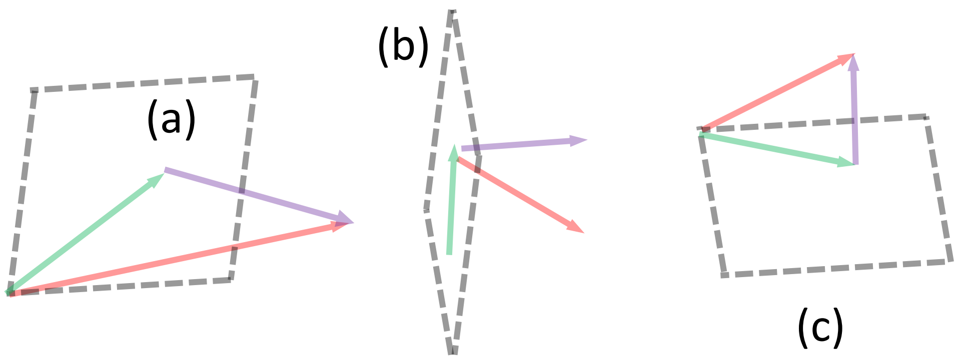
Stress tensor ($\boldsymbol{\sigma}$) on that location (●) linearly transforms the normal $\boldsymbol{n}$ to the traction vector $\boldsymbol{t}$

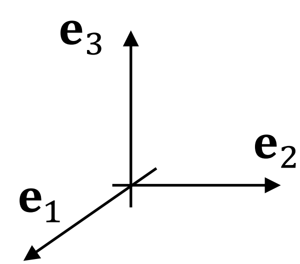
$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \boldsymbol{\sigma} \cdot \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \qquad \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

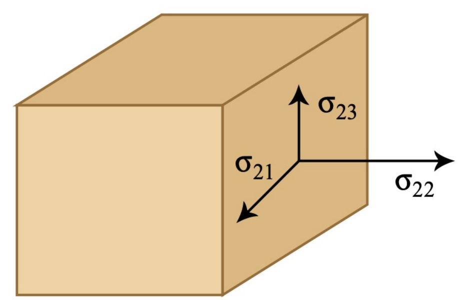
Mathematical description (both algebraic and geometrical)











각 면에 '누워서' (tangential) 작용하는 성분을 다시 $\mathbf{e}_i$에 projection 시켜 shear term은 2개, normal component는 1개가 각 면마다 있음.

Let's say by $\boldsymbol{t_n} = \frac{d\boldsymbol{f}}{dS}$

Here, we denote traction vector $\boldsymbol{t}$ acting on infinitesimal surface dS with the normal $\boldsymbol{n}$ as $\boldsymbol{t_n}$

$\boldsymbol{t}_{\mathbf{e}_i}$?

Traction vector $\boldsymbol{t}$ acting on infinitesimal surface dS with the basis normal $\mathbf{e}_i$ (note that i is a free index, it could be 1 or 2 or 3)

Three traction vectors on three planes

$$\boldsymbol{t}_{\mathbf{e}_1} = (\sigma_{11}, \sigma_{12}, \sigma_{13})$$
$$\boldsymbol{t}_{\mathbf{e}_2} = (\sigma_{11}, \sigma_{12}, \sigma_{13})$$
$$\boldsymbol{t}_{\mathbf{e}_3} = (\sigma_{11}, \sigma_{12}, \sigma_{13})$$

$$\boldsymbol{t} = \boldsymbol{\sigma} \cdot \boldsymbol{n} \qquad t_i = \sigma_{ij} n_j$$

$$\boldsymbol{t}_{\mathbf{e}_1} = \boldsymbol{\sigma} \cdot \mathbf{e}_1 \ ?$$

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$\begin{bmatrix} (\boldsymbol{t}_{\mathbf{e}_1})_1 \\ (\boldsymbol{t}_{\mathbf{e}_1})_2 \\ (\boldsymbol{t}_{\mathbf{e}_1})_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} (\mathbf{e}_1)_1 \\ (\mathbf{e}_1)_2 \\ (\mathbf{e}_1)_3 \end{bmatrix}$$

$$\begin{bmatrix} (\boldsymbol{t}_{\mathbf{e}_1})_1 \\ (\boldsymbol{t}_{\mathbf{e}_1})_2 \\ (\boldsymbol{t}_{\mathbf{e}_1})_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} (\boldsymbol{t}_{\mathbf{e}_1})_1 \\ (\boldsymbol{t}_{\mathbf{e}_1})_2 \\ (\boldsymbol{t}_{\mathbf{e}_1})_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{31} \end{bmatrix}$$

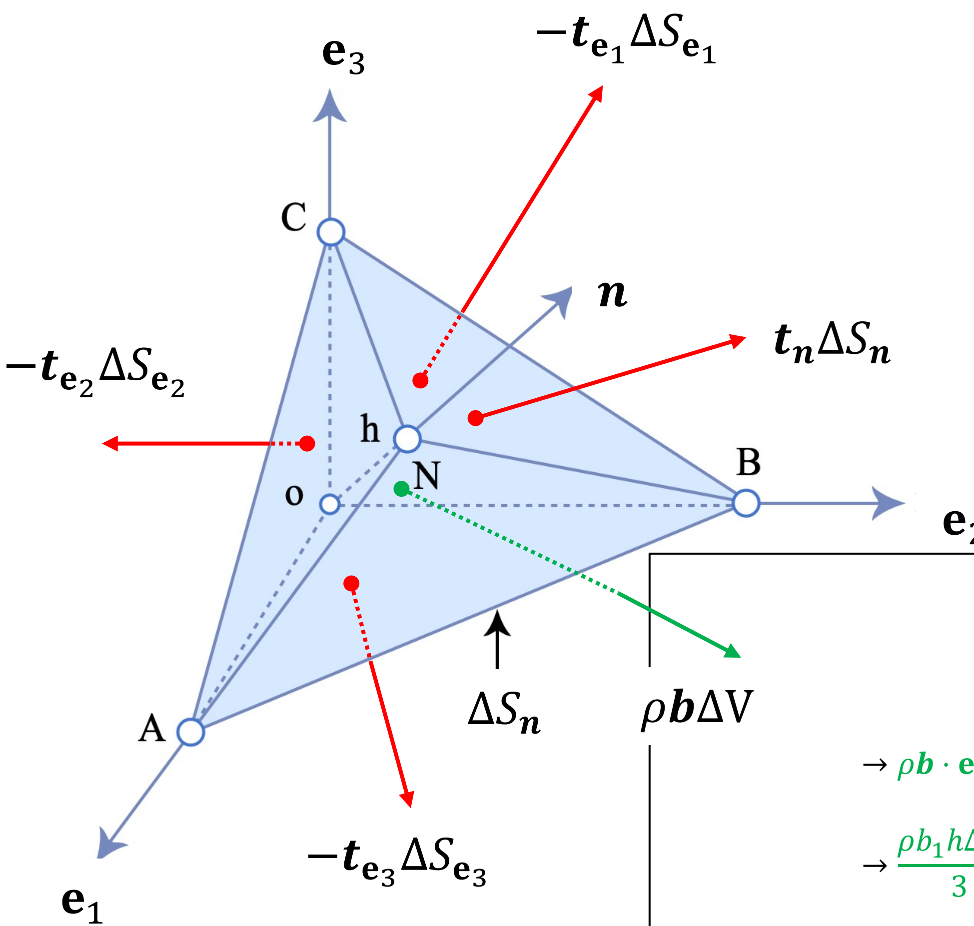
Caution: $\sigma_{ij} = \sigma_{ji}$

Cauchy stress tensor

Augustin-Louis Cauchy

- Causchy's idea: Traction vectors on three independent (perpendicular) planes pertaining to a point would suffice to provide the stress state of that point.
- Components of traction vectors on each of these three planes provide us the Cauchy stress components, which is sufficient to describe any arbitrary stress state. (No more than three planes!)
- While there are a number of strain measures …

Cauchy's Tetrahedron (사면체)



- $\mathbf{e}_i$: basis vectors (i is free)
- $\boldsymbol{t}_{\mathbf{e}_i}$: Traction acting on the plane $\perp$ $\mathbf{e}_i$
- $\Delta S_{\boldsymbol{n}}$: Area of A,B,C 삼각형
- $\Delta S_{\mathbf{e}_i}$: surface of the plane $\perp$ $\mathbf{e}_i$, which can be obtained by $\Delta S_{\mathbf{e}_i} = \Delta S_{\boldsymbol{n}}\, \boldsymbol{n} \cdot \mathbf{e}_i = \Delta S_{\boldsymbol{n}}\, n_i$ (Notice n_i is not in bold-face)
- b_i is the body force
- $m = \frac{1}{3}\rho h \Delta S_{\boldsymbol{n}}$

$$\boldsymbol{t} = \boldsymbol{\sigma} \cdot \boldsymbol{n}$$
$$t_i = \sigma_{ij} n_j$$
$$\boldsymbol{t}_{\mathbf{e}_1} = \boldsymbol{\sigma} \cdot \mathbf{e}_1$$

Force balance along $\mathbf{e}_1$?

Check $F_1 = m a_1$ (net F_1)

$$\mathbf{e}_1 \cdot \sum \boldsymbol{F} = \frac{1}{3}\rho h \Delta S_{\boldsymbol{n}} \boldsymbol{a} \cdot \mathbf{e}_1$$

$$\rightarrow \rho \boldsymbol{b} \cdot \mathbf{e}_1 \Delta V + (\boldsymbol{t}_{\boldsymbol{n}} \Delta S_{\boldsymbol{n}}) \cdot \mathbf{e}_1 + \left(\sum_i -\boldsymbol{t}_{\mathbf{e}_i} \Delta S_{\mathbf{e}_i} \right) \cdot \mathbf{e}_1 = \frac{1}{3}\rho h \Delta S_{\boldsymbol{n}} a_1$$

$$\rightarrow \frac{\rho b_1 h \Delta S_{\boldsymbol{n}}}{3} + \boldsymbol{t}_{\boldsymbol{n}} \cdot \mathbf{e}_1 \Delta S_{\boldsymbol{n}} - \Delta S_{\boldsymbol{n}} \left\{ \sum_i \boldsymbol{t}_{\mathbf{e}_i} n_i \right\} \cdot \mathbf{e}_1 = \frac{1}{3}\rho h \Delta S_{\boldsymbol{n}} a_1$$

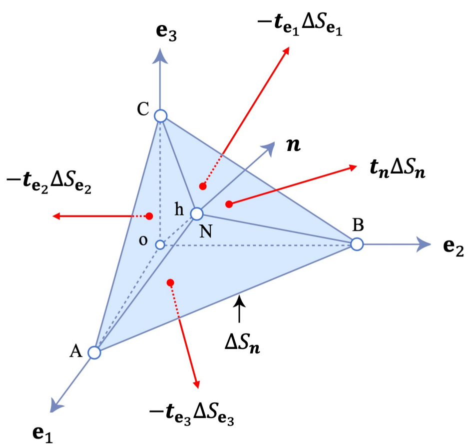
$$(LHS) \rightarrow \frac{\rho b_1 h \Delta S_{\boldsymbol{n}}}{3} + \boldsymbol{t}_{\boldsymbol{n}} \cdot \mathbf{e}_1 \Delta S_{\boldsymbol{n}} - \Delta S_{\boldsymbol{n}} \left\{ \sum_i n_i \boldsymbol{\sigma} \cdot \mathbf{e}_i \right\} \cdot \mathbf{e}_1$$

$$(LHS) \rightarrow \frac{\rho b_1 h \Delta S_{\boldsymbol{n}}}{3} + \boldsymbol{t}_{\boldsymbol{n}} \cdot \mathbf{e}_1 \Delta S_{\boldsymbol{n}} - \Delta S_{\boldsymbol{n}} \{ n_1 \boldsymbol{\sigma} \cdot \mathbf{e}_1 + n_2 \boldsymbol{\sigma} \cdot \mathbf{e}_2 + n_3 \boldsymbol{\sigma} \cdot \mathbf{e}_3 \} \cdot \mathbf{e}_1$$

$$(LHS) \rightarrow \frac{\rho b_1 h \Delta S_{\boldsymbol{n}}}{3} + \boldsymbol{t}_{\boldsymbol{n}} \cdot \mathbf{e}_1 \Delta S_{\boldsymbol{n}} - \Delta S_{\boldsymbol{n}} \{ n_1 \boldsymbol{\sigma} \cdot \mathbf{e}_1 \cdot \mathbf{e}_1 + n_2 \boldsymbol{\sigma} \cdot \mathbf{e}_2 \cdot \mathbf{e}_1 + n_3 \boldsymbol{\sigma} \cdot \mathbf{e}_3 \cdot \mathbf{e}_1 \}$$

$$F_1 = m a_1 \rightarrow \frac{\rho b_1 h \Delta S_{\boldsymbol{n}}}{3} + \boldsymbol{t}_{\boldsymbol{n}} \cdot \mathbf{e}_1 \Delta S_{\boldsymbol{n}} - \Delta S_{\boldsymbol{n}} \{ n_1 \sigma_{11} + n_2 \sigma_{21} + n_3 \sigma_{31} \} = \frac{1}{3}\rho h \Delta S_{\boldsymbol{n}} a_1$$

Cauchy's **Tetrahedron** (사면체)



- $\mathbf{e}_i$: basis vectors
- $\boldsymbol{t}_{\mathbf{e}_i}$: Traction acting on the plane $\perp$ $\mathbf{e}_i$
- $\Delta S_{\boldsymbol{n}}$: Area of A,B,C 삼각형
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- b_i is the body force
- $m = \frac{1}{3}\rho h \Delta S_{\boldsymbol{n}}$

Force balance along $\mathbf{e}_1$?

Check $F_1 = m a_1$ (net F_1)

$$F_1 = m a_1 \rightarrow \frac{\rho b_1 h \Delta S_{\boldsymbol{n}}}{3} + \boldsymbol{t}_{\boldsymbol{n}} \cdot \mathbf{e}_1 \Delta S_{\boldsymbol{n}} - \Delta S_{\boldsymbol{n}}\{n_1\sigma_{11} + n_2\sigma_{21} + n_3\sigma_{31}\} = \frac{1}{3}\rho h \Delta S_{\boldsymbol{n}} a_1$$

With $h \rightarrow 0$

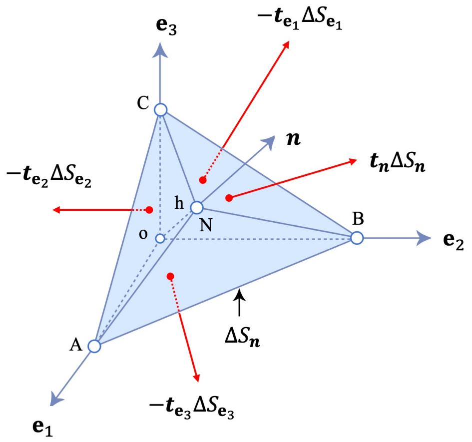
$$\rightarrow \boldsymbol{t}_{\boldsymbol{n}} \cdot \mathbf{e}_1 - \{n_1\sigma_{11} + n_2\sigma_{21} + n_3\sigma_{31}\} = \frac{1}{3}\rho h a_1$$

$$\boldsymbol{t}_{\boldsymbol{n}} \cdot \mathbf{e}_1 = \sum_j \sigma_{j1} n_j$$

Let's notice that $\boldsymbol{t}_{\boldsymbol{n}} \cdot \mathbf{e}_1 = (\boldsymbol{t}_{\boldsymbol{n}})_1$ and maybe it's better for us to shorten the notation $(\boldsymbol{t}_{\boldsymbol{n}})_1$ simply to t_1

Cauchy's **Tetrahedron** (사면체) (Advanced)

A case without body force



Force equilibrium:

$$\boldsymbol{t}_{\boldsymbol{n}}\Delta S_{\boldsymbol{n}} - \sum_i \boldsymbol{t}_{\mathbf{e}_i}\Delta S_{\mathbf{e}_i} = 0$$

Remember: $S_{\mathbf{e}_i} = \Delta S_{\boldsymbol{n}}\, \boldsymbol{n}\cdot\mathbf{e}_i = \Delta S_{\boldsymbol{n}}\, n_i$

$$\int_V \nabla\phi dV = \int_S \boldsymbol{n}\phi dS$$

- Say, ϕ is a quantity that is transferable (like heat, momentum, solute in solution etc.), transfer-in and -out should be met if equilibrium;
- In equlibrium (such as heat equilibrium or force equilibrium), $\phi = const.$
- Therefore, ($\boldsymbol{n}_S$ denotes normal of surface S)

$$\int_V \nabla\phi dV = \int_S \boldsymbol{n}_S\phi dS \to 0 = \int_S \boldsymbol{n}_S\phi dS \to 0 = \int_S \boldsymbol{n}_S dS$$

- For the case of this tetrahedron we have

$$\int_S \boldsymbol{n}_S dS = 0 \to \boldsymbol{n}\Delta S_{\boldsymbol{n}} - \mathbf{e}_1\Delta S_{\mathbf{e}_1} - \mathbf{e}_2\Delta S_{\mathbf{e}_2} - \mathbf{e}_3\Delta S_{\mathbf{e}_3} = 0$$

$$\to \boldsymbol{n}\Delta S_{\mathbf{n}} = \sum_i \mathbf{e}_i\Delta S_{\mathbf{e}_i} \to (\boldsymbol{n}\Delta S_{\mathbf{n}})\cdot\mathbf{e}_k = \mathbf{e}_k\cdot\sum_i \mathbf{e}_i\Delta S_{\mathbf{e}_i}$$

$$\to n_k\Delta S_{\boldsymbol{n}} = \sum_i \mathbf{e}_k\cdot\mathbf{e}_i\Delta S_{\mathbf{e}_i} = \Delta S_{\mathbf{e}_k} \to n_i\Delta S_{\boldsymbol{n}} = \Delta S_{\mathbf{e}_i}$$

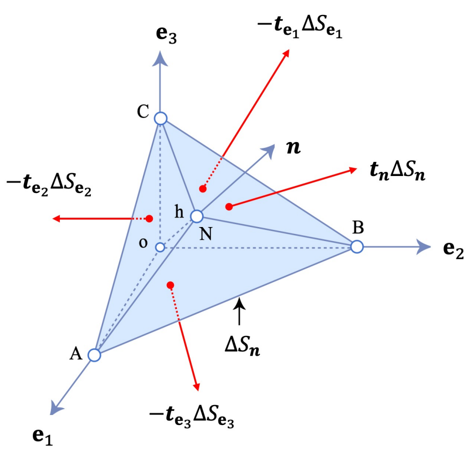
$$\to \text{Force equilibrum: } \boldsymbol{t}_{\boldsymbol{n}}\Delta S_{\boldsymbol{n}} - \sum_i \boldsymbol{t}_{\mathbf{e}_i} n_i\Delta S_{\boldsymbol{n}} = 0 \to \boldsymbol{t}_{\boldsymbol{n}} - \sum_i \boldsymbol{t}_{\mathbf{e}_i} n_i = 0$$

$$\boldsymbol{t}_{\boldsymbol{n}} = \sum_i \boldsymbol{t}_{\mathbf{e}_i} n_i \to \boldsymbol{t}_{\boldsymbol{n}} = \left(\boldsymbol{t}_{\mathbf{e}_1} n_1 + \boldsymbol{t}_{\mathbf{e}_2} n_2 + \boldsymbol{t}_{\mathbf{e}_3} n_3\right)$$

$$\to \boldsymbol{t}_{\boldsymbol{n}} = \left\{(\boldsymbol{n}\cdot\mathbf{e}_1)\boldsymbol{t}_{\mathbf{e}_1} + (\boldsymbol{n}\cdot\mathbf{e}_2)\boldsymbol{t}_{\mathbf{e}_2} + (\boldsymbol{n}\cdot\mathbf{e}_3)\boldsymbol{t}_{\mathbf{e}_3}\right\}$$

Cauchy's Tetrahedron (사면체) (Advanced)

A case without body force



$$\boldsymbol{t}_{\boldsymbol{n}} = \left\{(\boldsymbol{n}\cdot\mathbf{e}_1)\boldsymbol{t}_{\mathbf{e}_1} + (\boldsymbol{n}\cdot\mathbf{e}_2)\boldsymbol{t}_{\mathbf{e}_2} + (\boldsymbol{n}\cdot\mathbf{e}_3)\boldsymbol{t}_{\mathbf{e}_3}\right\}$$

$$\{\boldsymbol{t}_{\boldsymbol{n}}\}_i = \left\{(\boldsymbol{n}\cdot\mathbf{e}_1)\boldsymbol{t}_{\mathbf{e}_1} + (\boldsymbol{n}\cdot\mathbf{e}_2)\boldsymbol{t}_{\mathbf{e}_2} + (\boldsymbol{n}\cdot\mathbf{e}_3)\boldsymbol{t}_{\mathbf{e}_3}\right\}_i \text{ ; note the free index } i$$

$$\{\boldsymbol{t}_{\boldsymbol{n}}\}_i = (\boldsymbol{n}\cdot\mathbf{e}_1)\left\{\boldsymbol{t}_{\mathbf{e}_1}\right\}_i + (\boldsymbol{n}\cdot\mathbf{e}_2)\left\{\boldsymbol{t}_{\mathbf{e}_2}\right\}_i + (\boldsymbol{n}\cdot\mathbf{e}_3)\left\{\boldsymbol{t}_{\mathbf{e}_3}\right\}_i$$

Notice that both $\{\boldsymbol{t}_{\boldsymbol{n}}\}_i$ and $\left\{\boldsymbol{t}_{\mathbf{e}_k}\right\}_i$ (k is free index) are 'scalars' – they for each index represent each component, as in below,

$$c\boldsymbol{a} = ca_i\mathbf{e}_i = a_i(c\boldsymbol{e}_i)$$

so that

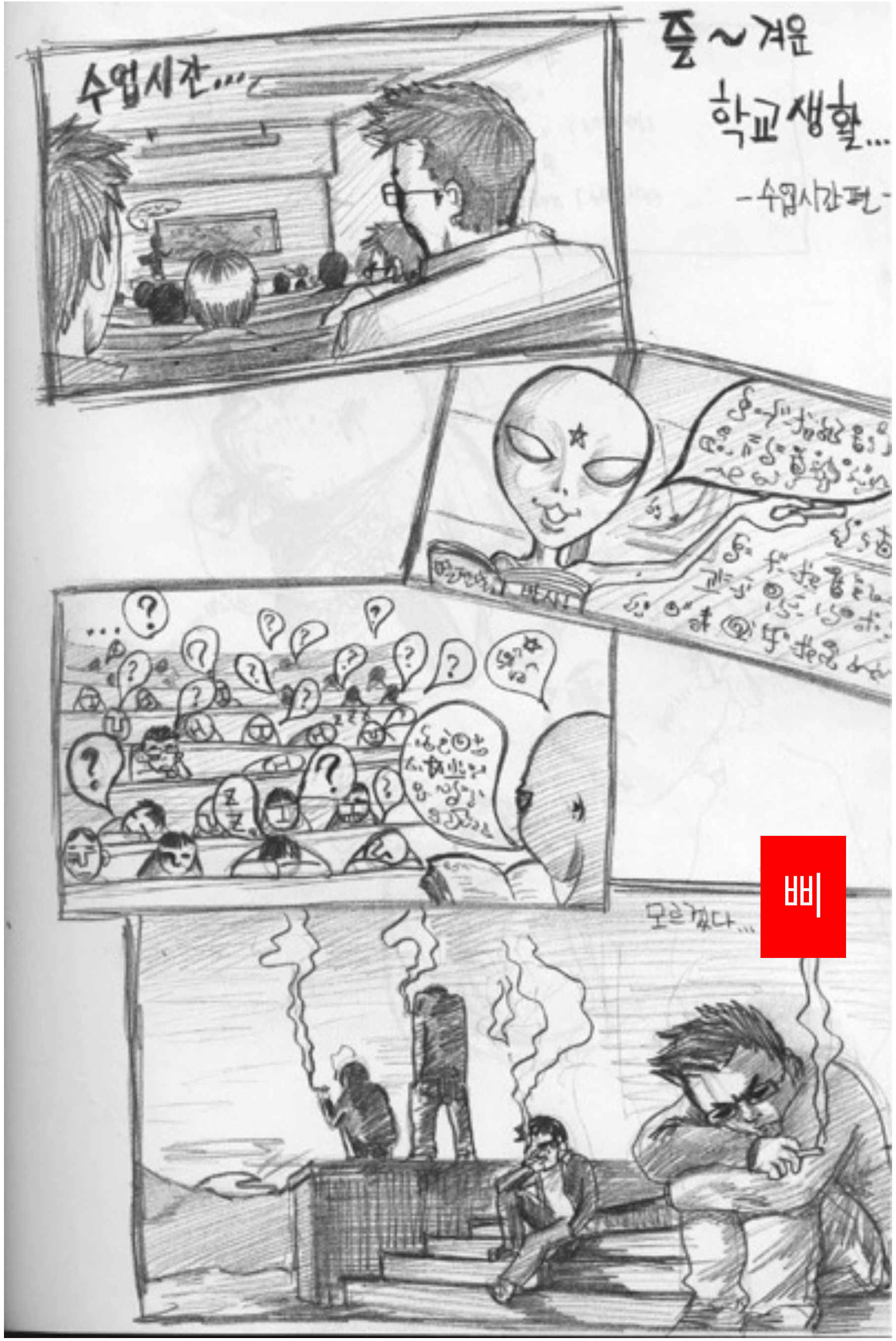
$$\{\boldsymbol{t}_{\boldsymbol{n}}\}_i = (\boldsymbol{n})\cdot\mathbf{e}_1\left\{\boldsymbol{t}_{\mathbf{e}_1}\right\}_i + (\boldsymbol{n})\cdot\mathbf{e}_2\left\{\boldsymbol{t}_{\mathbf{e}_2}\right\}_i + (\boldsymbol{n})\cdot\mathbf{e}_3\left\{\boldsymbol{t}_{\mathbf{e}_3}\right\}_i$$
$$= \boldsymbol{n}\cdot\left\{\mathbf{e}_1\boldsymbol{t}_{\mathbf{e}_1} + \mathbf{e}_2\boldsymbol{t}_{\mathbf{e}_2} + \mathbf{e}_3\boldsymbol{t}_{\mathbf{e}_3}\right\}_i$$

$$\boldsymbol{t}_{\boldsymbol{n}}\cdot\mathbf{e}_i = \boldsymbol{n}\cdot\left(\mathbf{e}_1\boldsymbol{t}_{\mathbf{e}_1} + \mathbf{e}_2\boldsymbol{t}_{\mathbf{e}_2} + \mathbf{e}_3\boldsymbol{t}_{\mathbf{e}_3}\right)\cdot\mathbf{e}_i$$

Force equilibrium:

$$\boldsymbol{t}_{\boldsymbol{n}}\Delta S_{\boldsymbol{n}} - \sum_i \boldsymbol{t}_{\mathbf{e}_i}\Delta S_{\mathbf{e}_i} = 0$$

Remember: $S_{\mathbf{e}_i} = \Delta S_{\boldsymbol{n}}\,\boldsymbol{n}\cdot\mathbf{e}_i = \Delta S_{\boldsymbol{n}}\,n_i$



"I'm a college professor, Jason. You need to ask someone else if you want advice about the real world."

Stress tensor is symmetric

$$\sigma_{ij} = \sigma_{ji}$$

Therefore, the below

$$\boldsymbol{t_n} \cdot \mathbf{e}_1 = \sum_j \sigma_{j1} n_j$$

could be equivalently written as:

$$(\boldsymbol{t_n})_1 = \sum_j \sigma_{1j} n_j$$

And, therefore, by making use Indicial and Einstein notations,

$$(\boldsymbol{t_n})_i = \sigma_{ij} n_j$$

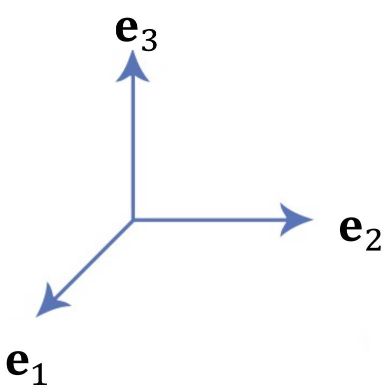
Proof of symmetric stress tensor

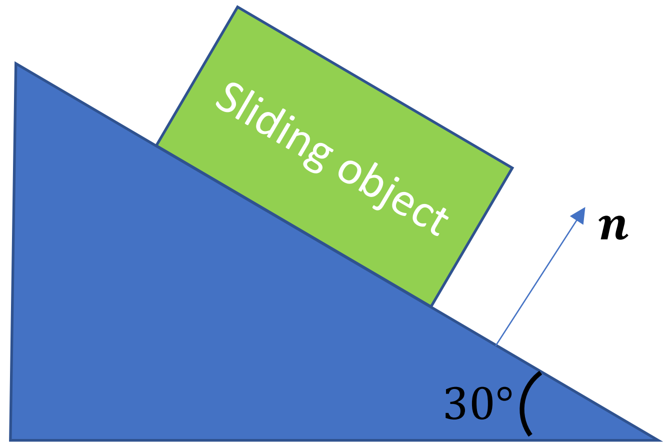
The same linear transformation applies to **strain** tensor

$\boldsymbol{\varepsilon} \cdot \boldsymbol{n}$ gives you the 'stretched' vector.

If you know the meaning of dot product, you'll understand $(\boldsymbol{\varepsilon} \cdot \boldsymbol{n}) \cdot \boldsymbol{n}$ gives you a fractional change in length expected along $\boldsymbol{n}$ expected as an outcome of $\boldsymbol{\varepsilon}$.

Object sliding downhill





Points under the object is under the stress of

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -25 & 0 \\ 0 & 0 & 40 \end{bmatrix}$$

Q: What is the traction vector acting on the plane?

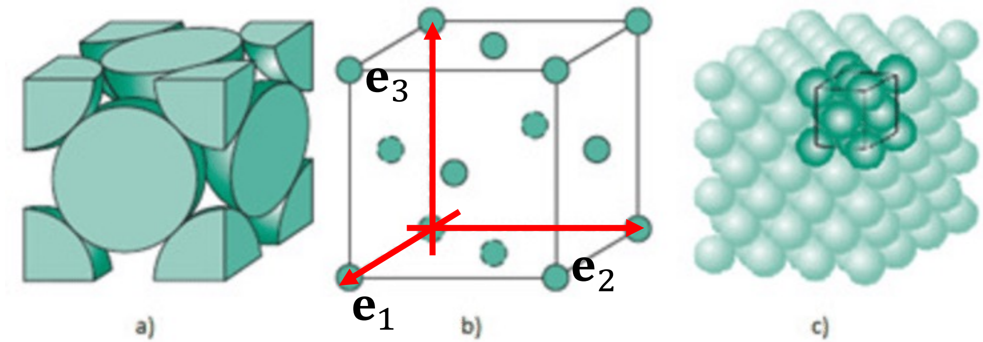
solution:

$$\boldsymbol{n} = [0, \cos 60°, \sin 60°] = \left[0,0.5,\frac{\sqrt{3}}{2}\right]$$

$$\boldsymbol{t} = \boldsymbol{\sigma} \cdot \boldsymbol{n} \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & -25 & 0 \\ 0 & 0 & -40 \end{bmatrix} \begin{bmatrix} 0 \\ 0.5 \\ 0.866 \end{bmatrix} = [0, -12.5, -34.6]$$

Examples.

Suppose a single crystal Ni (in FCC) is subjected to a traction field of (100, 10, 10) MPa.



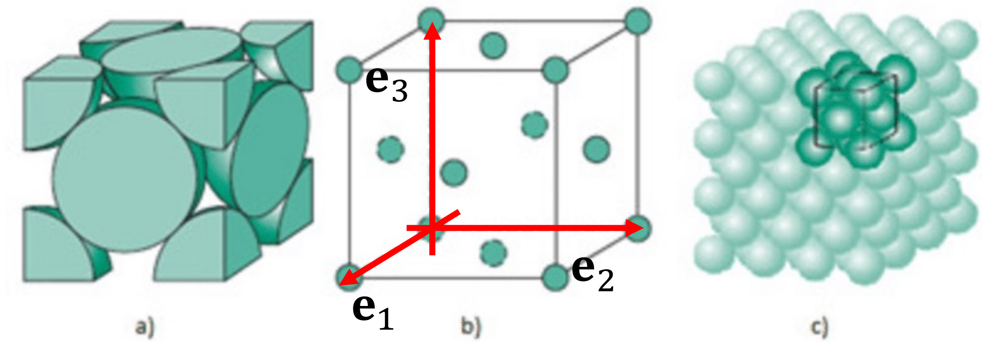
Calculate the traction vector along (111) plane normal

Solution). Since, say, $\boldsymbol{n} = \frac{(1,1,1)}{|(1,1,1)|} = \frac{1}{\sqrt{3}}(1,1,1)$. The component of traction vector (give as $\boldsymbol{t} = (100,10,10)$), its component along n is obtained by projecting t along n such that

$$\boldsymbol{t} \cdot \boldsymbol{n} = \frac{120}{\sqrt{3}}$$

Examples.

Suppose a single crystal Ni (in FCC) is subjected to a traction field of (100, 10, 10) MPa.



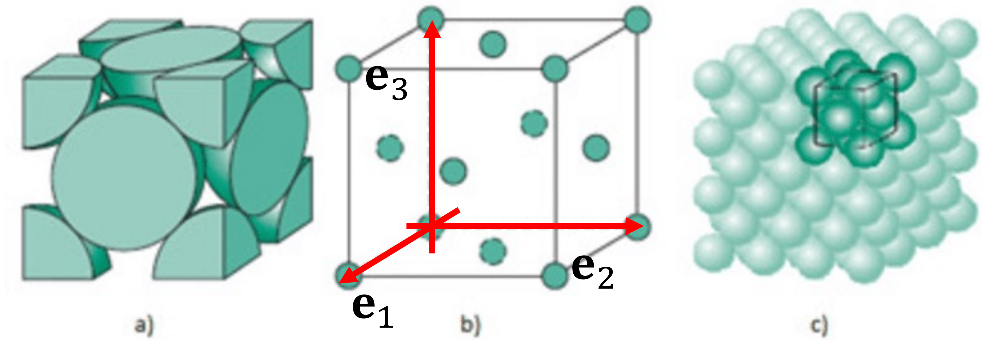
Calculate the traction vector along $(1\bar{1}1)$ plane normal

(Solution). Since, say, $\boldsymbol{n} = \frac{(1,1,1)}{|(1,1,1)|} = \frac{1}{\sqrt{3}}(1,\bar{1},1)$. The component of traction vector (given as $\boldsymbol{t} = (100,10,10)$) along n is obtained by projecting t along n such that

$$\boldsymbol{t} \cdot \boldsymbol{n} = \frac{100}{\sqrt{3}}$$

Resolved shear stress

Suppose a single crystal Fe (in BCC) is subjected a stress of $\begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 0 \end{bmatrix}$



Calculate the resolved shear stress acting on system $(011)[1\bar{1}1]$

(Solution). Since, say, $\boldsymbol{n} = \frac{(0,1,1)}{|(0,1,1)|} = \frac{1}{\sqrt{2}}(0,1,1)$. The traction vector (denoted as $\boldsymbol{t}$) on plane $\boldsymbol{n}$ is

$$\boldsymbol{t} = \frac{\sqrt{2}}{2}\begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} 20 \\ 0 \\ 0 \end{bmatrix}$$

In order to obtain $\boldsymbol{t}$ component in the slip direction (denoted as $\boldsymbol{b}$), use dot-product, such that $\tau = \boldsymbol{t} \cdot \boldsymbol{b} = \frac{\sqrt{2}}{2}\frac{1}{\sqrt{3}}\begin{bmatrix} 20 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \frac{\sqrt{2}}{2}\frac{1}{\sqrt{3}} \cdot 20 \cong 8.16$

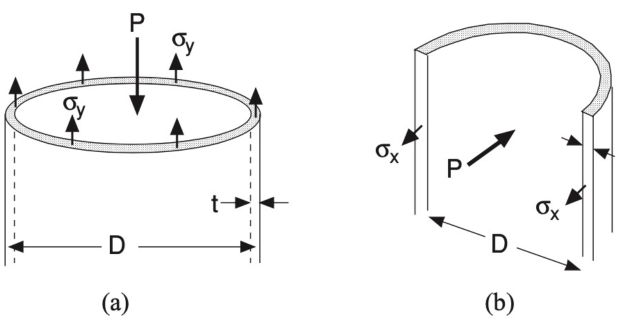
Generalized Schmid's law

$$\tau = (\boldsymbol{\sigma} \cdot \boldsymbol{n}) \cdot \boldsymbol{b} = \boldsymbol{b} \cdot (\boldsymbol{\sigma} \cdot \boldsymbol{n}) = \boldsymbol{b} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n} = b_i \sigma_{ij} n_j$$

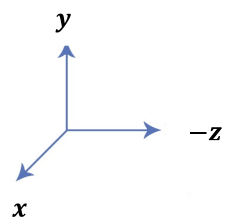
Force and moment balances

- Net force acting on any portion of a body should be zero.

- Externally applied forces (die, punch and so forth) should be balanced by internal forces – attractions and repulsions between atoms.



* 길이가 L 이고, 지름이 D 그리고 두께가 t인 파이프의 내부가 압력 P를 겪는다.



길이 방향이 normal인 면에 작용하는 응력 성분 σ_y은, 이러한 force balance를 활용하여 구할 수 있다.

$$\sum_{\text{source}} F_y^{\text{each source}} = F_y^{\text{pipe pressure}} + F_y^{\text{internal stress}} = -\frac{P\pi D^2}{4} + \pi D t \sigma_y = 0$$

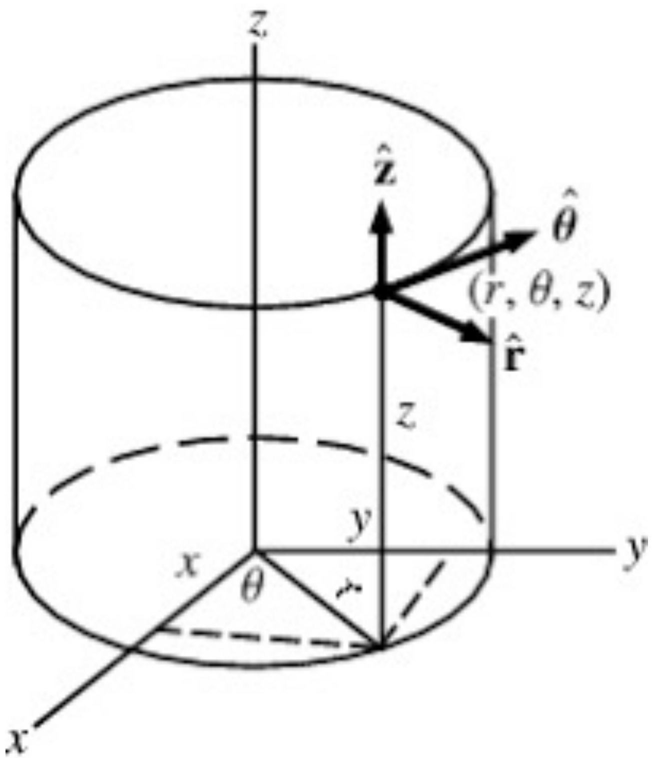
$$\rightarrow \pi D t \sigma_y = \frac{P\pi D^2}{4} \rightarrow \sigma_y = \frac{PD}{4t}$$

$$\sum_{\text{source}} F_x^{\text{each source}} = -PDL + 2\sigma_x t L = 0 \rightarrow \sigma_x = \frac{PD}{2t}$$

Force and moment balances

- $\boldsymbol{m} = \boldsymbol{F} \times \boldsymbol{x} \;\; \rightarrow \boldsymbol{m}_k = f_i x_j \mathbf{e}_i \times \mathbf{e}_j = f_i x_j \epsilon_{ijk} \mathbf{e}_k$

$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \parallel \hat{\boldsymbol{r}}, \widehat{\boldsymbol{\theta}}, \hat{\boldsymbol{z}}$



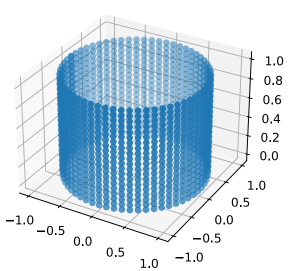
Shear stress

$\sigma_{r\theta}$

$$\boldsymbol{m}_3 = f_i x_j \epsilon_{ij3} \mathbf{e}_3 = (f_1 x_2 - f_2 x_1)\mathbf{e}_3$$

$$\boldsymbol{m}_3 = (t_1 A_r x_2 - t_2 A_\theta x_1)\mathbf{e}_3$$

$$= \left(\sigma_{1j} n_j^{(r)} A_r x_2 - \sigma_{2k} n_k^{(r)} A_\theta x_1\right)\mathbf{e}_3$$



$\sigma_{r\theta}$