Displacement and strain

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Outline

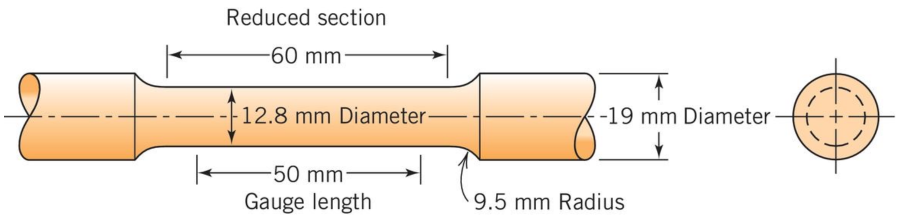
- 본 강의에서는 금속의 기계적 성질을 표현하는데 가장 중요한 요소인 응력과 변형률에 대해서 살펴본다.

- 응력과 변형률이 텐서(tensor)로 표현 되는 방법에 대해 알아보고 텐서의 기본적인 성질에 대해 알아보자.

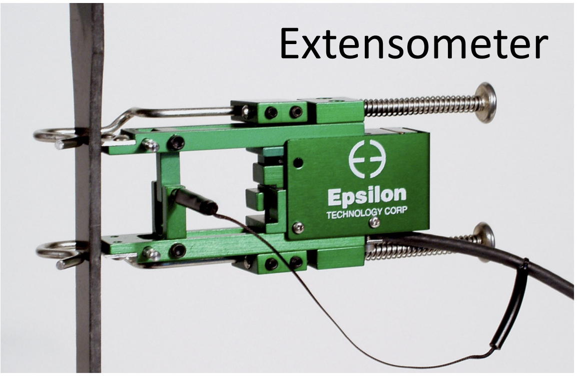
Tension tests

- (Uniaxial) tension test: the most common mechanical stress-strain test performed in **tension** (인장)
- 시편(specimen)은 주로 파괴(fracture)가 발생할 때가지 당겨진다.
- Dogbone 모양의 시편
- Load-cell: 시편에 가해진 force를 측정
- Extensometer: 시편의 길이 (elongation) 변화를 측정

Tensile tester



Load-cell



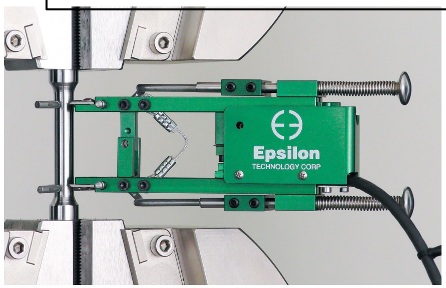
Extensometer

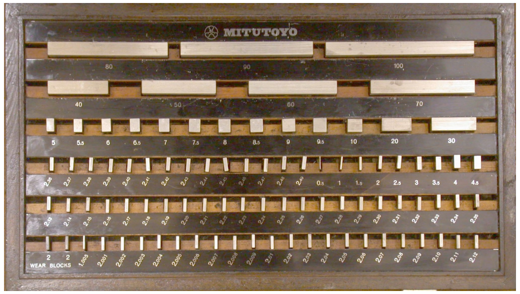
Images from Callister, Int. MSE

http://www.epsilontech.com/products/axial-extensometer-model-3542/

Measuring strains

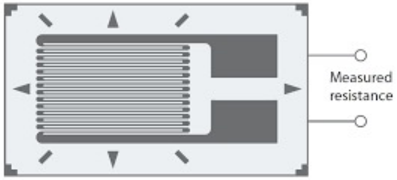
Measuring Relative motion of two points



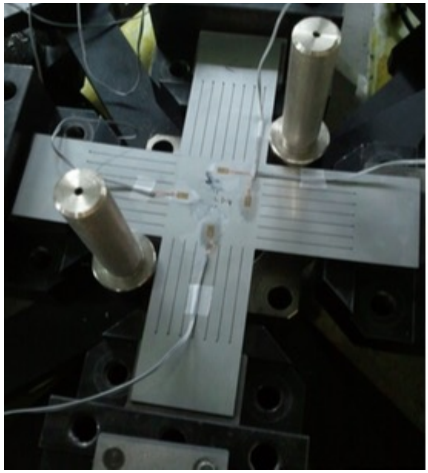


Mechanical extensometer and gauge blocks

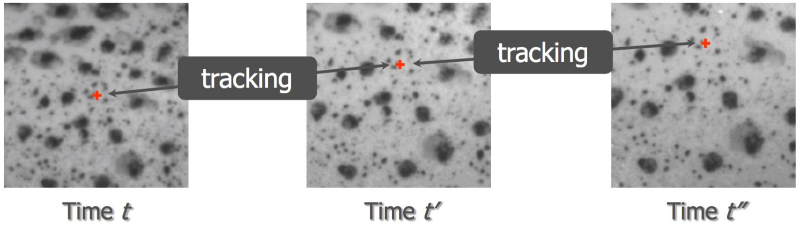
Strain pertaining to a smaller area (point)



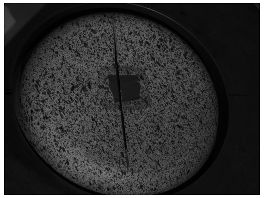
Strain gauge

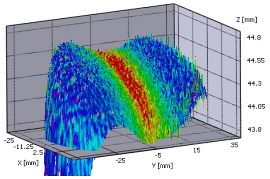


Strain field measurement (2D area)

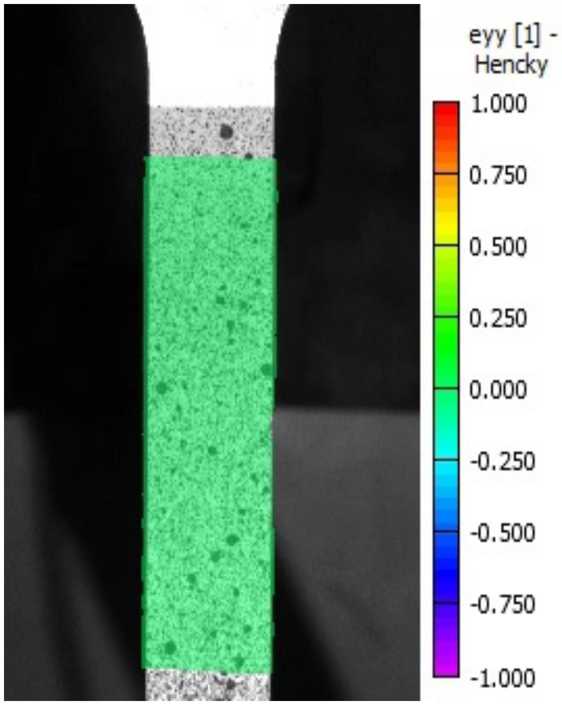


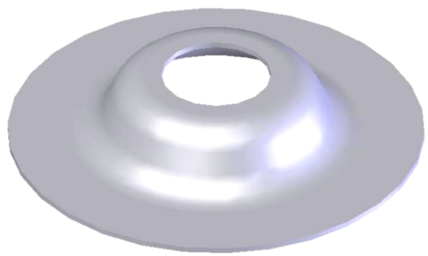
Digital image correlation by tracking displacement **field** $\boldsymbol{u}(X, Y)$

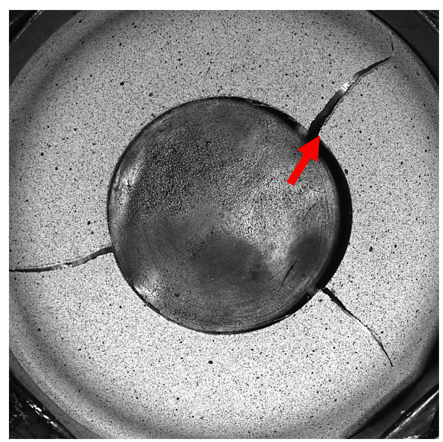


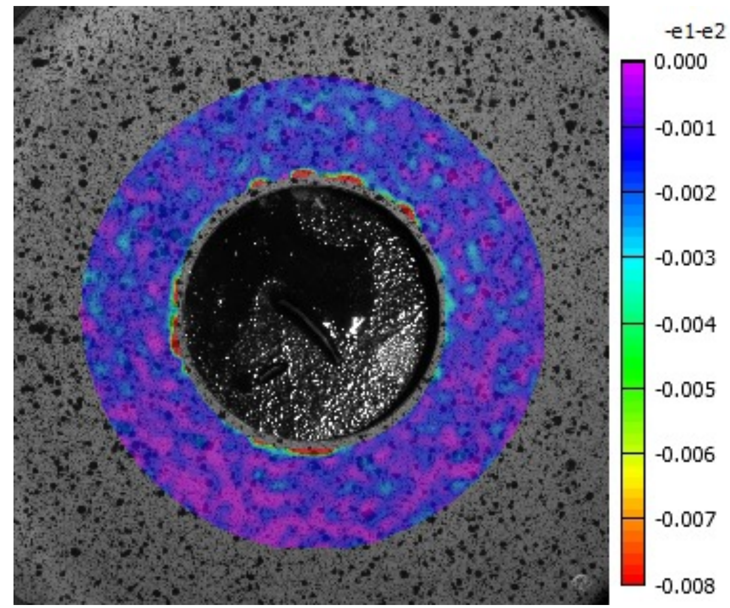


Digital image correlation









Questions

- 우리가 측정할 수 있는 것은 ‘변위(displacement)’이다. 변위를 가지고 재료가 얼마나 ’변형’이 되었는지를 알려줄 수 있는 지표를 얻을 수 있을까?

- 우리의 목표는 측정가능한 ‘변위’를 사용해 ‘변형’을 정량적으로 표현할 수 있는 ‘방법’을 찾는 것이다.

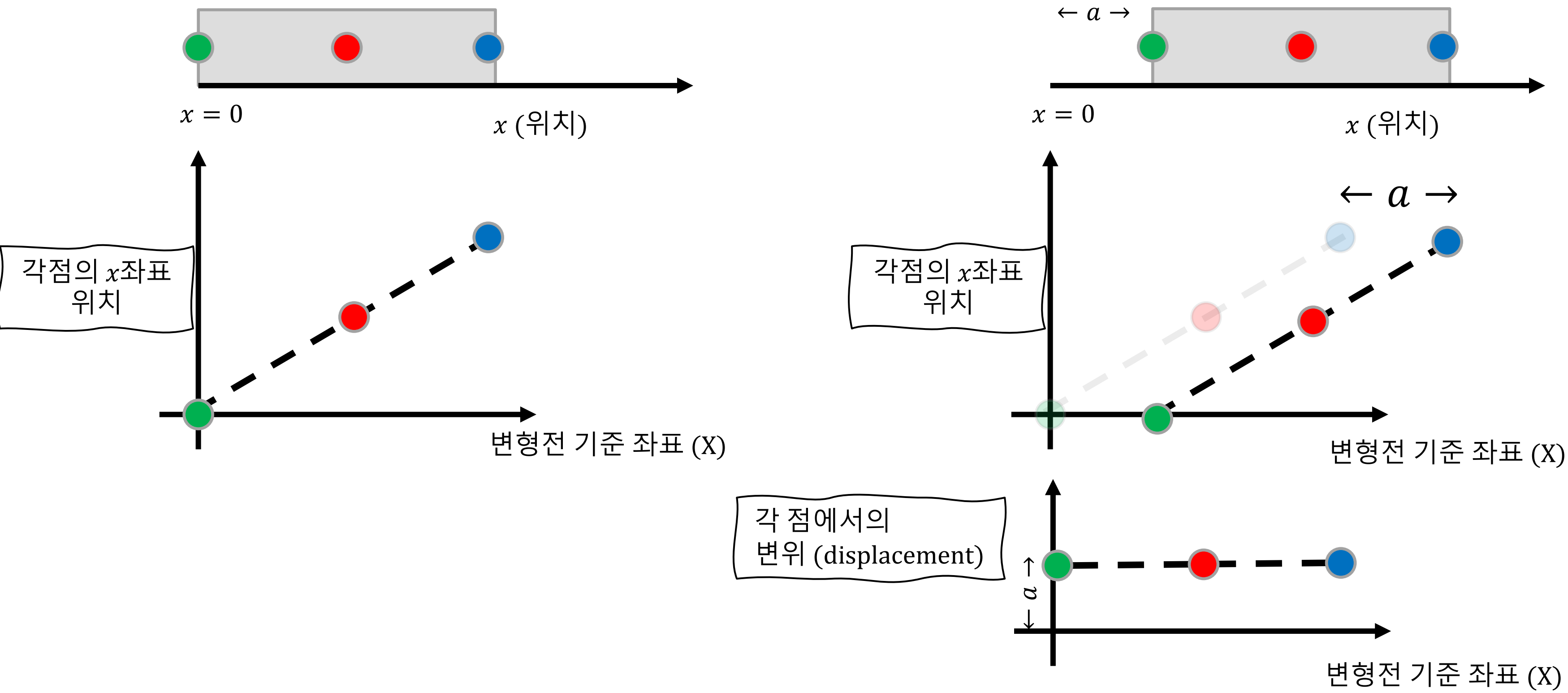
- 변위를 이용해 ‘변형률(strain)’을 얻는다. 변형률이 ‘변형’의 정량적 지표가 된다.
  - 변위가 무엇인지 엄밀한 정의 필요
  - 변위를 활용해 변형률을 정의하는 방법
    - Small strain theory
    - Finite strain theory

변위 (displacement)와 변형… (1)

- 물체의 ‘변형’을 우리가 측정할 수 있는 물리량으로써 나타내보자.
- 우리가 쉽게 측정할 수 있는 물리량은 물체의 각 지점의 ‘좌표’ (가령, x,y 축 좌표점)이다.
- 물체를 이루는 모든 점들의 ' 좌표’ 변화를 파악한다면, 물체의 ‘형태변화’를 알 수 있다.

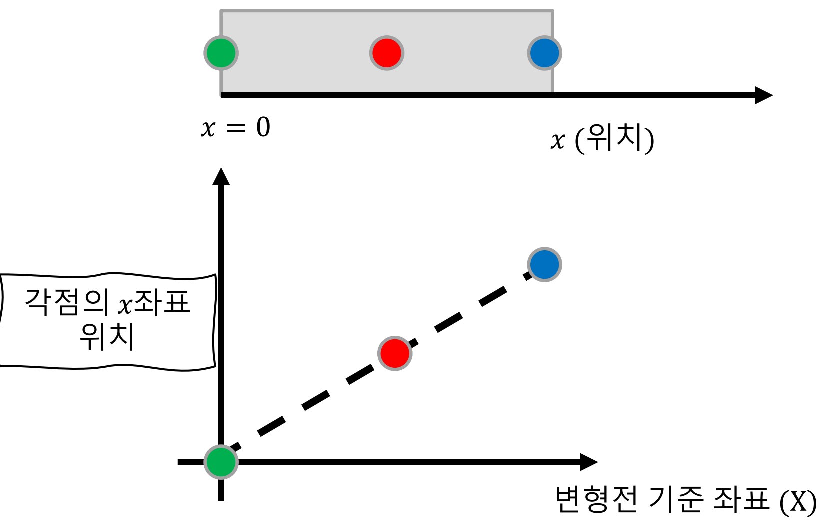
- 변위는 ‘field variable’
  - 따라서 공간 (X,Y)에 대한 함수로 표현가능.

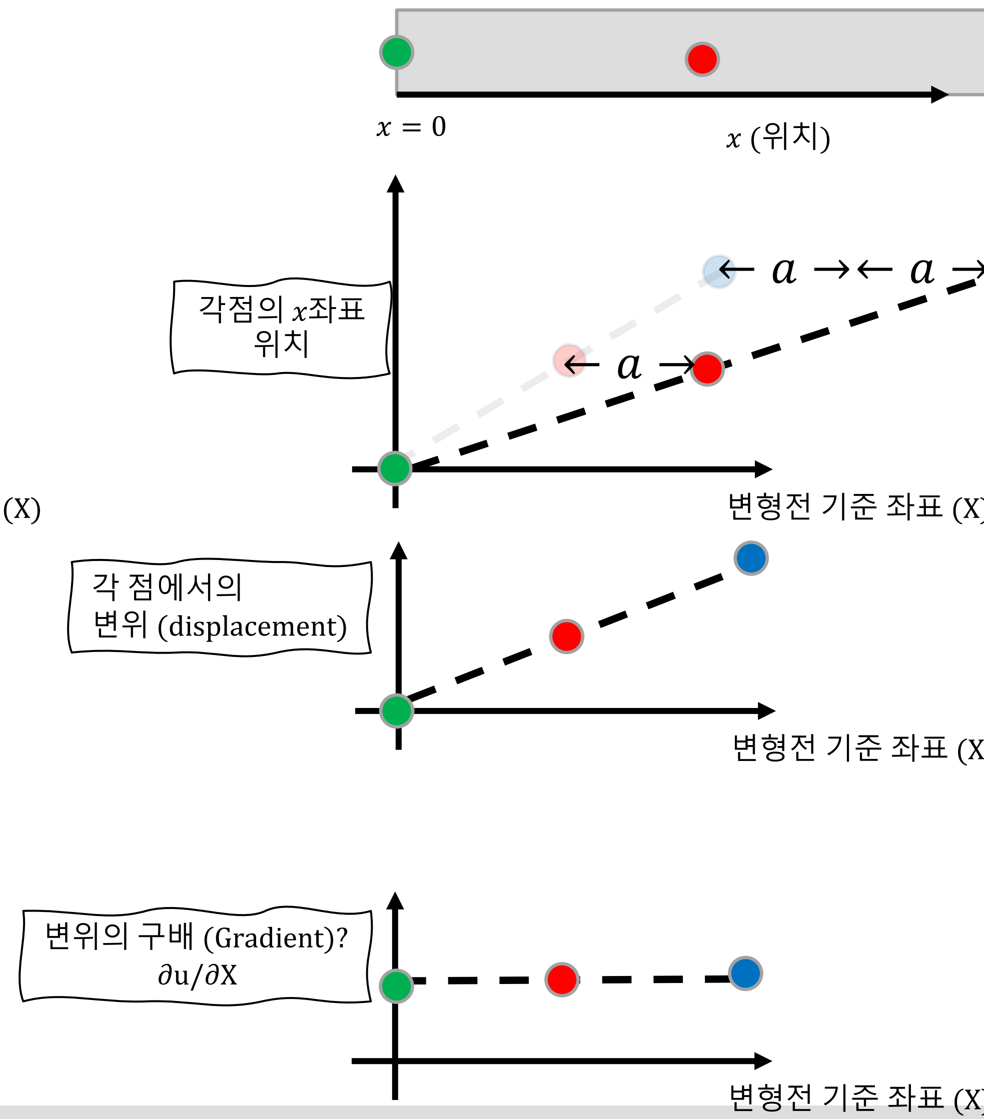
변위 (displacement)와 변형... (2)



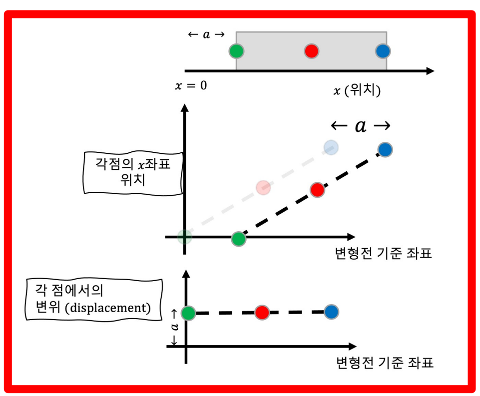
변위의 값과 변형과는 무관

변위 (displacement)와 변형… (3)





Previous slide



변위의 값과 변형과는 무관

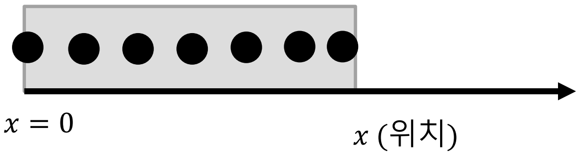
** 변위의 구배는 '변형전 기준 좌표' 사용.
참고로, 변형전 기준 좌표가 아니라 변형후의 좌표를 사용하는 경우도 존재한다.

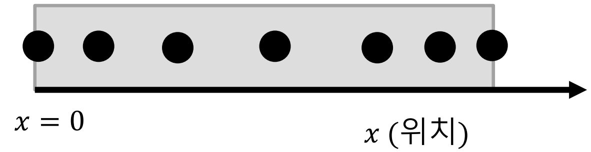
변위 (displacement)와 변형… (4)

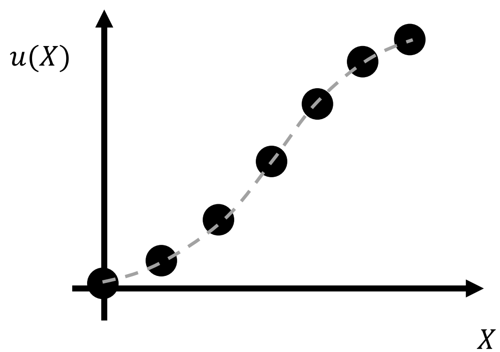
다음 중 물체의 ‘변형’을 나타내는 물리량으로 적절한 것은?

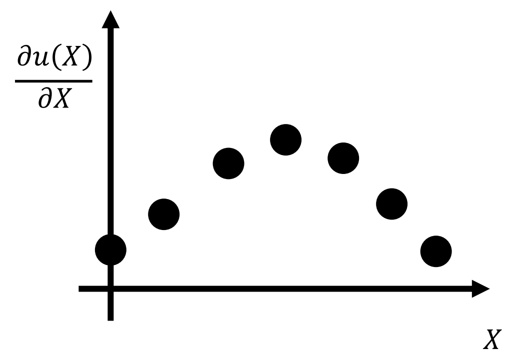
(1) 변위
(2) 물질점의 좌표
(3) 변위의 변화량
(4) 위치에 따라 달라지는 변위의 변화량

변위 (displacement)와 변형... (5-1)



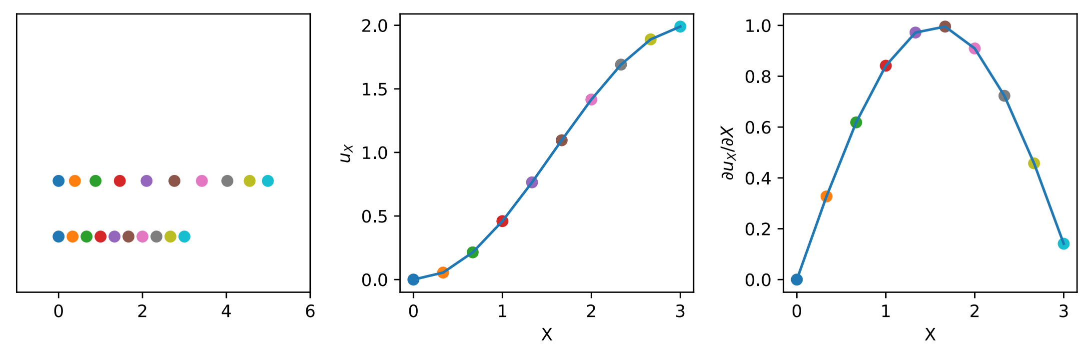






위와 같은 '운동'은 다음과 같은 변위(u)의 위치(X)에 대한 함수로 표현된다.

$$u(X) = -\cos(X)$$

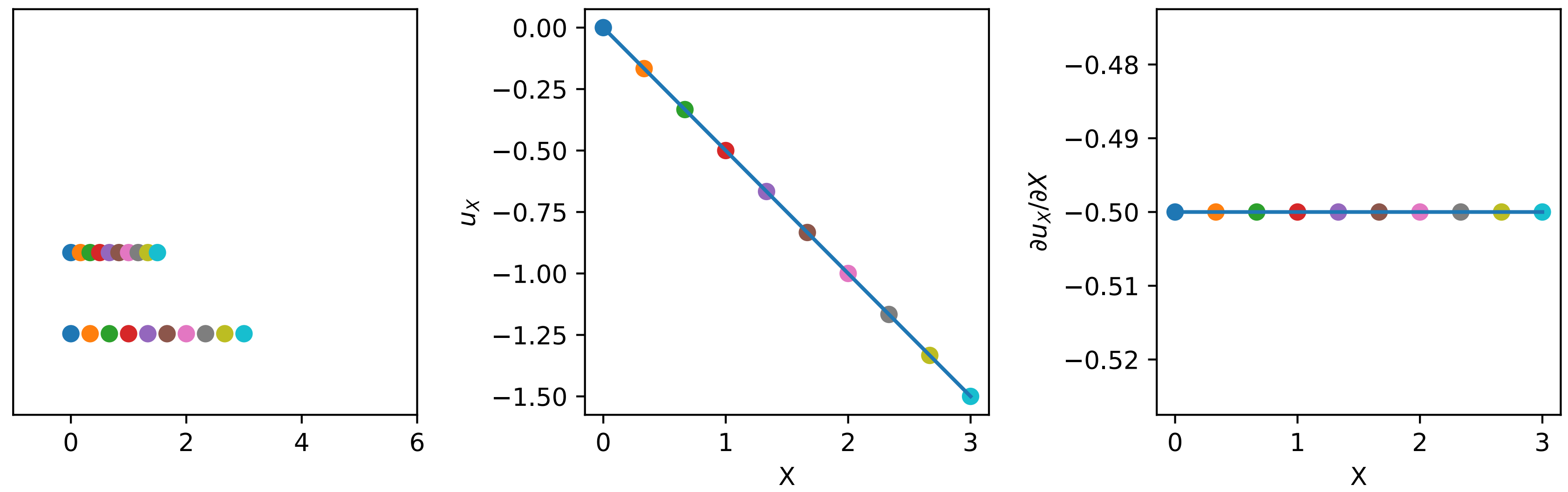


Q위의 운동은 균일한 변형으로 이어지나? (무엇을 근거로 판단하나?)

변위 (displacement)와 변형… (5-2)

Say, the displacement is a function of X such that

$$u(X) = -0.5X$$



Negative slope of displacement led to 'compression'
압축 변형이 발생하면, 그에 해당하는 변위 구배값이 음이다.

Q. 변위의 구배가 변형의 지표로 적절한가?

1차원에서의 변위 구배

- 변위가 0이 아니라면 (운동)이 존재. 모든 운동(motion)이 변형으로 이어지는 것은 아니다.
- 1차원공간의 1차원재료의 변위라면 스칼라 물리량만으로도 충분히 표현된다.
- **변위 구배**가 0이 아니라면 변형이 존재한다.

- 변위는 field variable (위치에 따라 달라 질 수 있는 물리량)으로 여기자.

위치 (position)

- 1차원 공간에서의 위치 (X)
- 2차원 공간에서의 위치 (X,Y) 혹은 (X_1, X_2) 혹은 $\boldsymbol{X}$
- 3차원 공간에서의 위치 (X,Y,Z) 혹은 (X_1, X_2, X_3) 혹은 $\boldsymbol{X}$

- 수업 자료에 때때로 (X,Y,Z)와 (X_1, X_2, X_3) 표기법이 섞여 사용되어 있습니다.

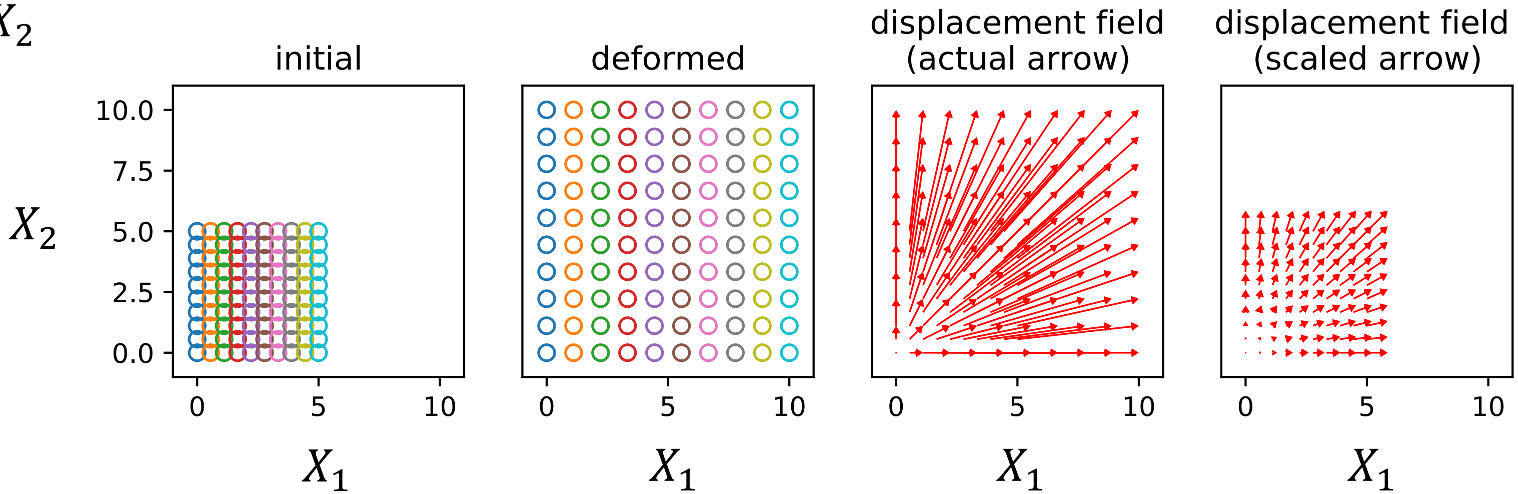
변위 (displacement)와 변형... (7)

- 2차원 공간(X_1, X_2) 에서의 변위

$$\boldsymbol{u} \equiv \boldsymbol{u}(X_1, X_2)$$

- Equi-biaxial deformation

$$u_{X_1} = 0.5X_1$$
$$u_{X_2} = 0.5X_2$$

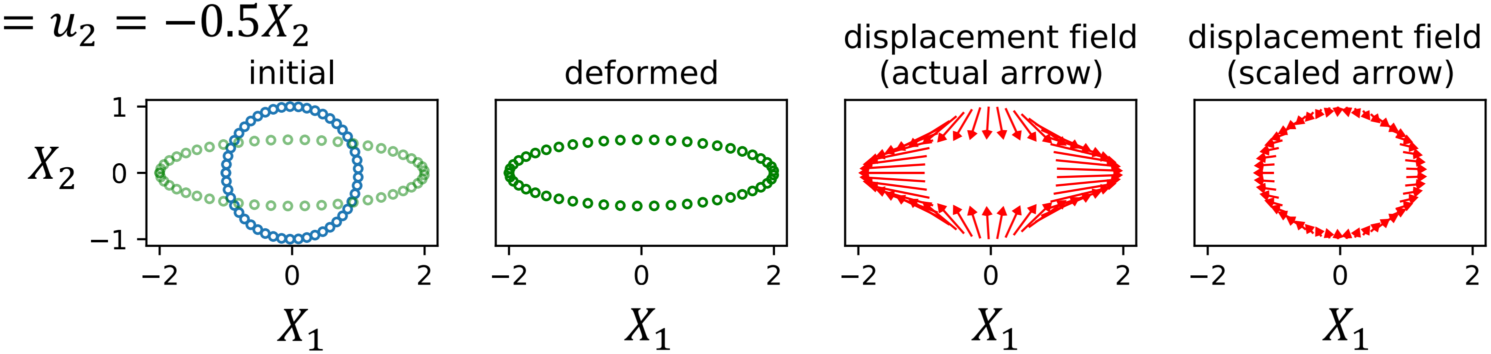


균일한 변형은, 변위구배가 좌표계와 상관없이 일정하다.

위 변형은, 균일한가?

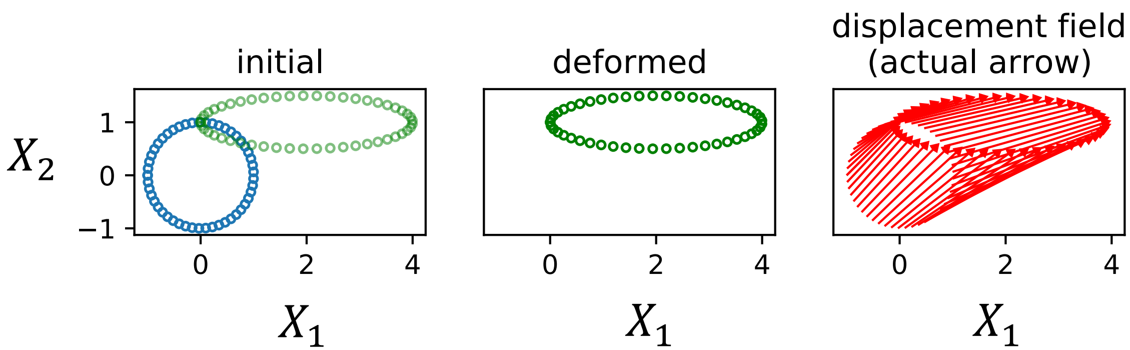
변위 (displacement)와 변형... (7)

$u_{X_1} = u_1 = X_1$
$u_{X_2} = u_2 = -0.5X_2$



Q. 아래의 변위는 위의 변위의 경우와 같은 **변형**을 일으킨다. 왜 그럴까?

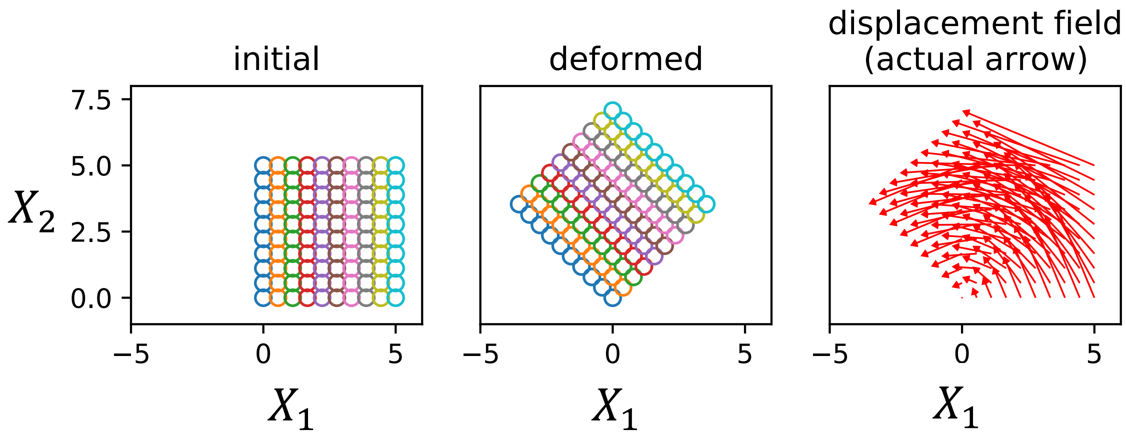
$$u_{X_1} = X_1 + 2$$
$$u_{X_2} = -0.5X_2 + 1$$



Not all high $\frac{\partial u_i}{\partial X_j}$ means high distortion

$$u_{X_1} = 0.707X_1 - 0.707X_2$$
$$u_{X_2} = 0.707X_1 + 0.707X_2$$

- leads to:



It was ***not deformed***; but instead, it simply rotated counter-clock-wise!

$$\begin{bmatrix} u_{X_1} \\ u_{Y_2} \end{bmatrix} = \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

Linear map of $[X_1, X_2]$ to $[u_{X_1}, u_{X_2}]$

From initial to final

- $\boldsymbol{x}^{final} = \boldsymbol{x}^{initial} + \boldsymbol{u}(\boldsymbol{x}^{initial})$

$$u_{X_1} = u_1 = X_1$$
$$u_{X_2} = u_2 = -0.5X_2$$

$$\begin{bmatrix} x_1^{final} \\ x_2^{final} \end{bmatrix} = \begin{bmatrix} x_1^{initial} \\ x_2^{initial} \end{bmatrix} + \begin{bmatrix} \frac{\partial u_1}{\partial X_1} & \frac{\partial u_1}{\partial X_2} \\ \frac{\partial u_2}{\partial X_1} & \frac{\partial u_2}{\partial X_2} \end{bmatrix} \begin{bmatrix} x_1^{initial} \\ x_2^{initial} \end{bmatrix}$$
$$= \begin{bmatrix} x_1^{initial} \\ x_2^{initial} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -0.5 \end{bmatrix} \begin{bmatrix} x_1^{initial} \\ x_2^{initial} \end{bmatrix}$$

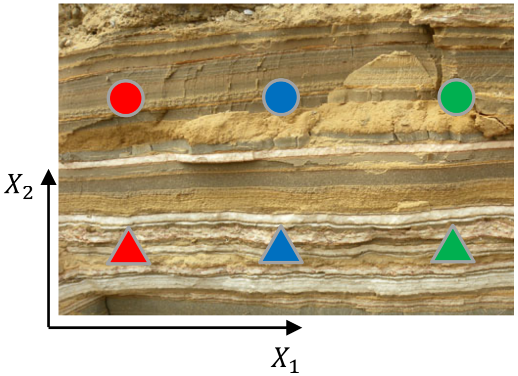
$$u_{ij} = \frac{\partial u_i}{\partial X_j}$$

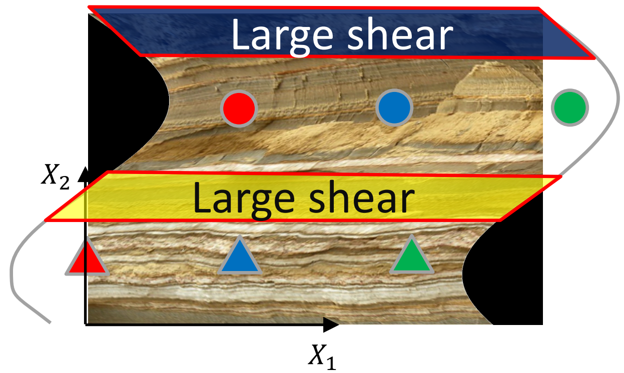
$$x_i^{final} = \left\{ \delta_{ij} + \frac{\partial u_i}{\partial X_j} \right\} x_j^{initial}$$

2차원에서의 변위 구배

- 변위가 0이 아니라면 (운동)이 존재. 모든 운동(motion)이 변형으로 이어지는 것은 아니다.
- 다차원(2차원 이상)에서 변위 구배가 0이 아니라고해서 변형이 존재하는 것은 아니다.
- 다차원 (2차원 이상)에서는 변위가 ‘벡터’ 물리량으로 표현된다.

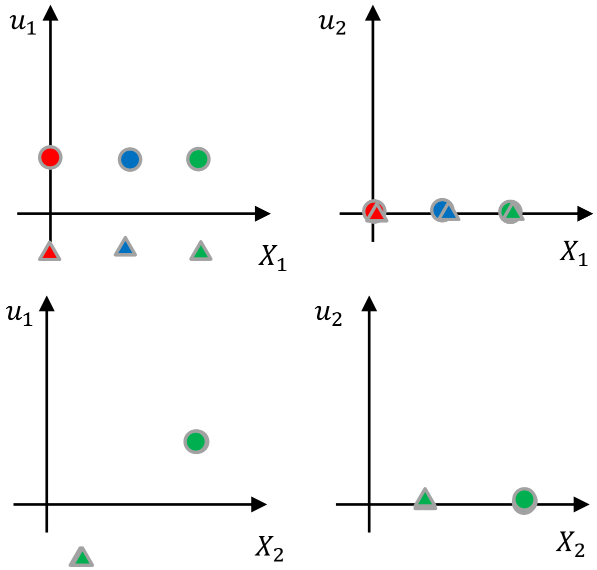
변위 (displacement)와 변형... (6)

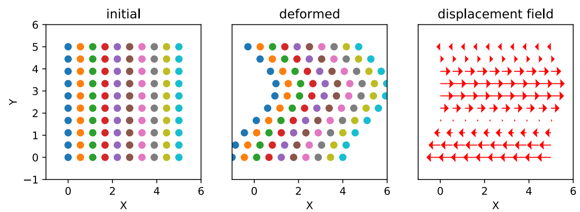


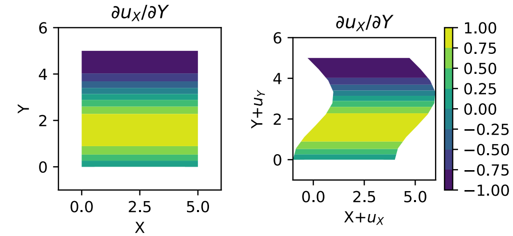


다음과 같이 지층 아래 방향으로 sine 곡선 형태 **변위**가 발생했다. 가장 크게 **변형**이 발생한 지점은 어디인가?

$$u_1 = -\cos(X_2)$$
$$u_2 = 0$$

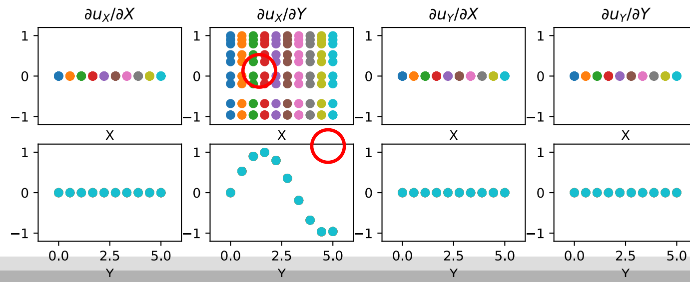






Eulerian

Lagrangian



shear

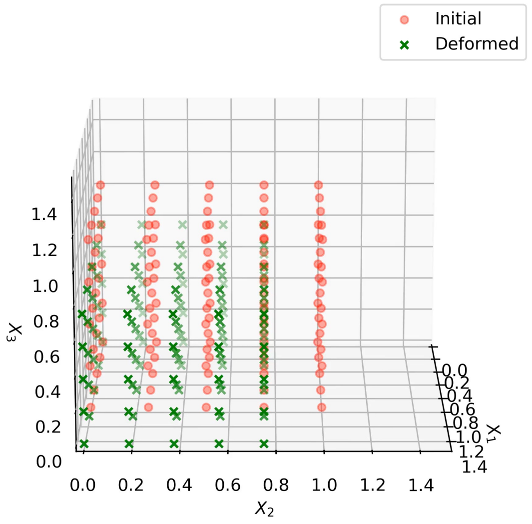
- 변위의 구배 여부를 살필 때, 관심있는 공간의 방향과 어긋난 방향 (예를 들어 수직 방향)으로의 변화 여부를 살필 필요가 있다.

- 즉, $\frac{\partial u_{X_1}}{\partial X_1}$ 뿐만 아니라 $\frac{\partial u_{X_1}}{\partial X_2}$ 도 살펴야 한다. 왜냐면, $\frac{\partial u_{X_1}}{\partial X_2}$ 값이 0이 아닐 때 ‘변형’ (특히 shear)이 나타날 수도 있으니까.

Small strain theory

Infinitesimal strain tensor $\boldsymbol{\varepsilon}$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)$$



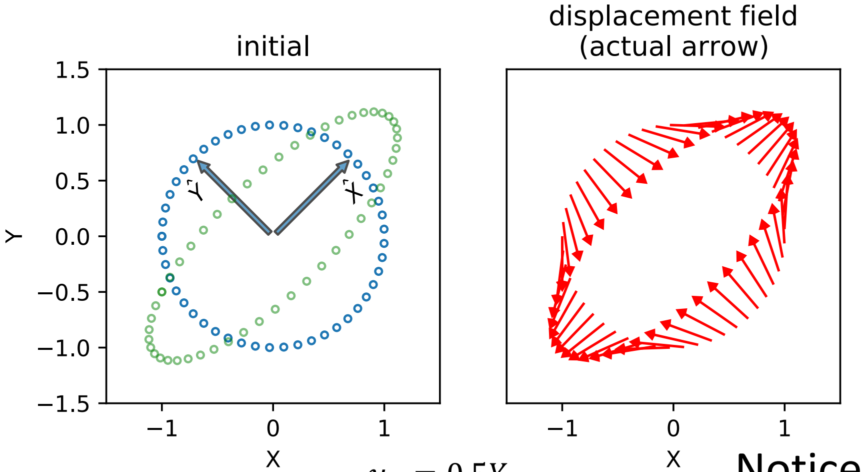
Example 1)

$$\boldsymbol{u}(\boldsymbol{X}) = (0.5X_1, -0.25X_2, -0.25X_3)$$

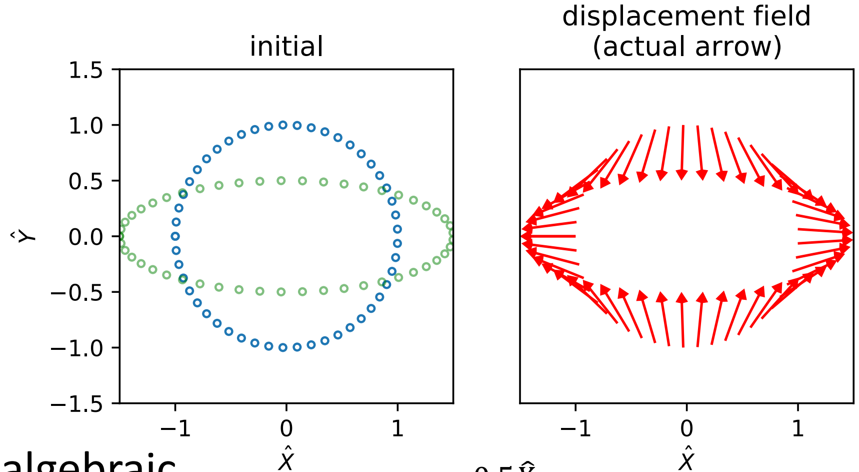
$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}? \quad \begin{bmatrix} \frac{\partial u_1}{\partial X_1} & \frac{\partial u_1}{\partial X_2} & \frac{\partial u_1}{\partial X_3} \\ \frac{\partial u_2}{\partial X_1} & \frac{\partial u_2}{\partial X_2} & \frac{\partial u_2}{\partial X_3} \\ \frac{\partial u_3}{\partial X_1} & \frac{\partial u_3}{\partial X_2} & \frac{\partial u_3}{\partial X_3} \end{bmatrix} \quad \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & -0.25 & 0 \\ 0 & 0 & -0.25 \end{bmatrix}$$

좌표계 선택의 임의성

- 좌표계는 좌표계를 선택하는 이의 편의로 정하기 마련이다. 중요한 것은 이렇게 임의로 정해진 좌표계의 선택이 물리 현상을 설명하는 방법에 영향을 끼쳐서는 안된다는 점이다.



$$u_X = 0.5Y$$
$$u_Y = 0.5X$$



$$u_{\hat{X}} = 0.5\hat{X}$$
$$u_{\hat{Y}} = -0.5\hat{Y}$$

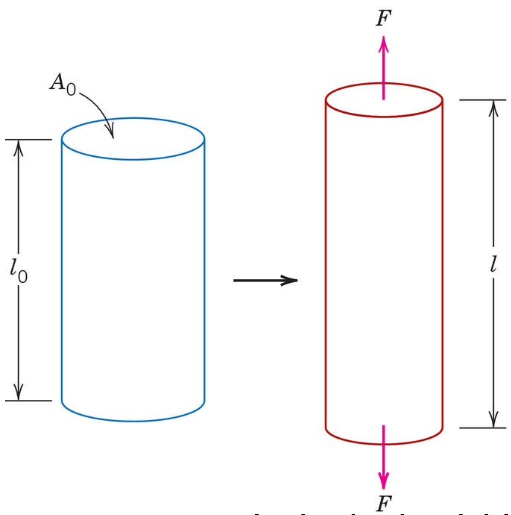
Notice the algebraic difference!

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \frac{1}{2}\left(\frac{\partial u_X}{\partial X} + \frac{\partial u_X}{\partial X}\right) & \frac{1}{2}\left(\frac{\partial u_X}{\partial Y} + \frac{\partial u_Y}{\partial X}\right) \\ \frac{1}{2}\left(\frac{\partial u_Y}{\partial X} + \frac{\partial u_X}{\partial Y}\right) & \frac{1}{2}\left(\frac{\partial u_Y}{\partial Y} + \frac{\partial u_Y}{\partial Y}\right) \end{bmatrix} = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}$$

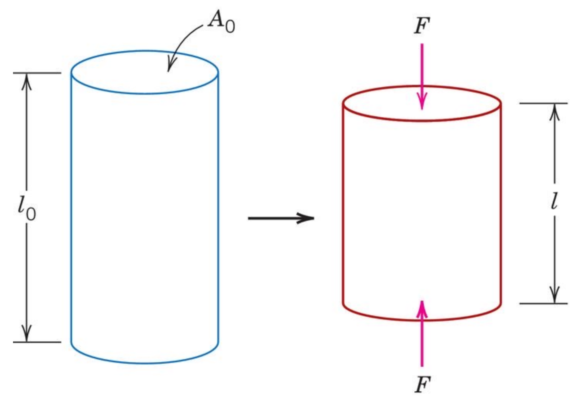
$$\boldsymbol{\varepsilon} = \begin{bmatrix} \frac{1}{2}\left(\frac{\partial u_{\hat{X}}}{\partial \hat{X}} + \frac{\partial u_{\hat{X}}}{\partial \hat{X}}\right) & \frac{1}{2}\left(\frac{\partial u_{\hat{X}}}{\partial \hat{Y}} + \frac{\partial u_{\hat{Y}}}{\partial \hat{X}}\right) \\ \frac{1}{2}\left(\frac{\partial u_{\hat{Y}}}{\partial \hat{X}} + \frac{\partial u_{\hat{X}}}{\partial \hat{Y}}\right) & \frac{1}{2}\left(\frac{\partial u_{\hat{Y}}}{\partial \hat{Y}} + \frac{\partial u_{\hat{Y}}}{\partial \hat{Y}}\right) \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & -0.5 \end{bmatrix}$$

Realize that the algebraic quantity is valid for a specific coordinate system and cannot convey the invariance of physical law on the choice of coordinate system by itself – requires something more profound (**the concept of tensor**)

구조물에 작용하는 하중의 종류 (from Callister textbook)

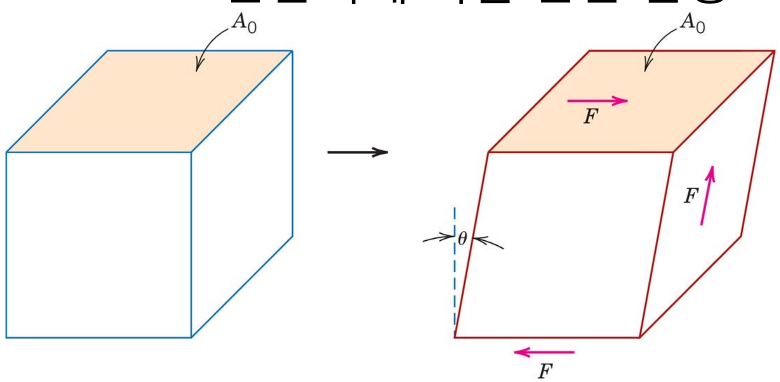


인장
(tension)

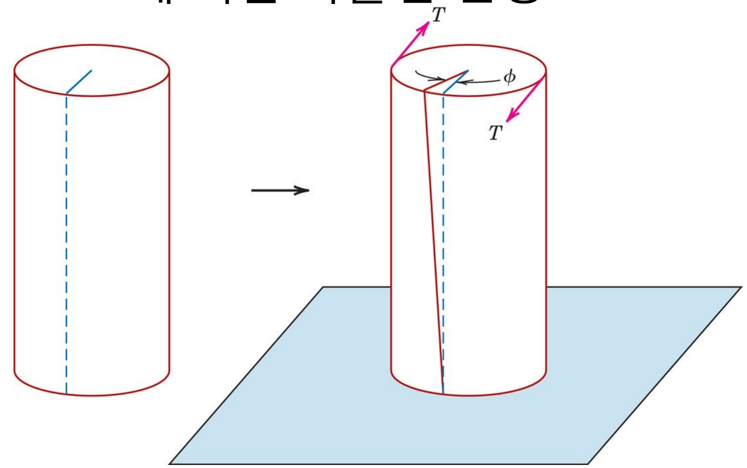


압축
(compression)

전단력에 의한 전단 변형



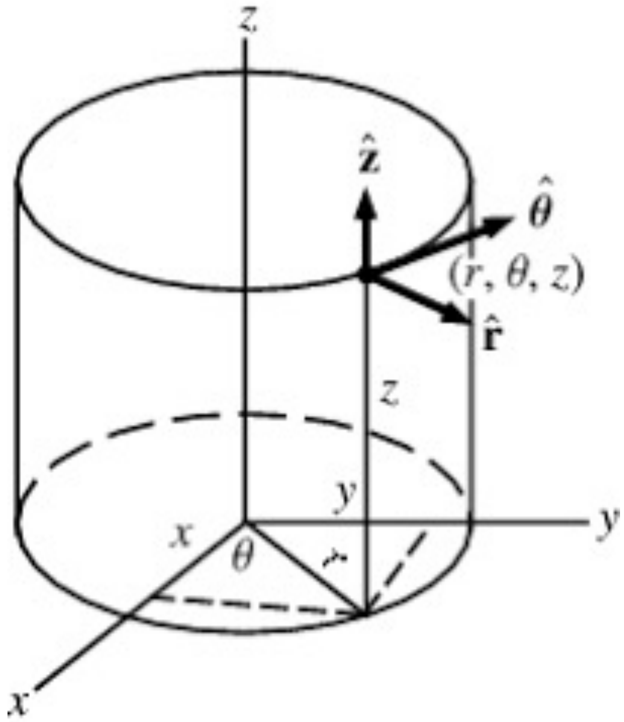
토크에 의한 비틀림 변형



Images from Callister, Int. MSE

Displacement field in cylindrical coordinates

$a\ point\ in\ cylindrical\ coodinates: (r, \theta, z)$



$$\boldsymbol{u} = \boldsymbol{u}(r, \theta, z)$$

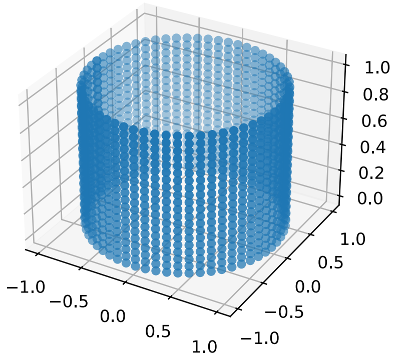
$$\nabla \mathbf{u} = \begin{bmatrix} \frac{\partial u_r}{\partial r} & \frac{\partial u_r}{\partial \theta} & \frac{\partial u_r}{\partial z} \\ \frac{\partial u_\theta}{\partial r} & \frac{\partial u_\theta}{\partial \theta} & \frac{\partial u_\theta}{\partial z} \\ \frac{\partial u_z}{\partial r} & \frac{\partial u_\theta}{\partial \theta} & \frac{\partial u_\theta}{\partial z} \end{bmatrix}$$

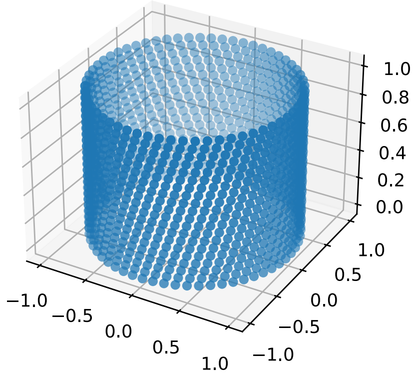
Displacement field in cylindrical coordinates

$$\nabla \mathbf{u} = \begin{bmatrix} \frac{\partial u_r}{\partial r} & \frac{\partial u_r}{\partial \theta} & \frac{\partial u_r}{\partial z} \\ \frac{\partial u_\theta}{\partial r} & \frac{\partial u_\theta}{\partial \theta} & \frac{\partial u_\theta}{\partial z} \\ \frac{\partial u_z}{\partial r} & \frac{\partial u_z}{\partial \theta} & \frac{\partial u_z}{\partial z} \end{bmatrix}$$

Below initial and deformed points follows from

$$\mathbf{u} = (u_r, u_\theta, u_z) = (0, 0.5z, 0)$$





$$\nabla \mathbf{u} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.5 \\ 0 & 0 & 0 \end{bmatrix}$$

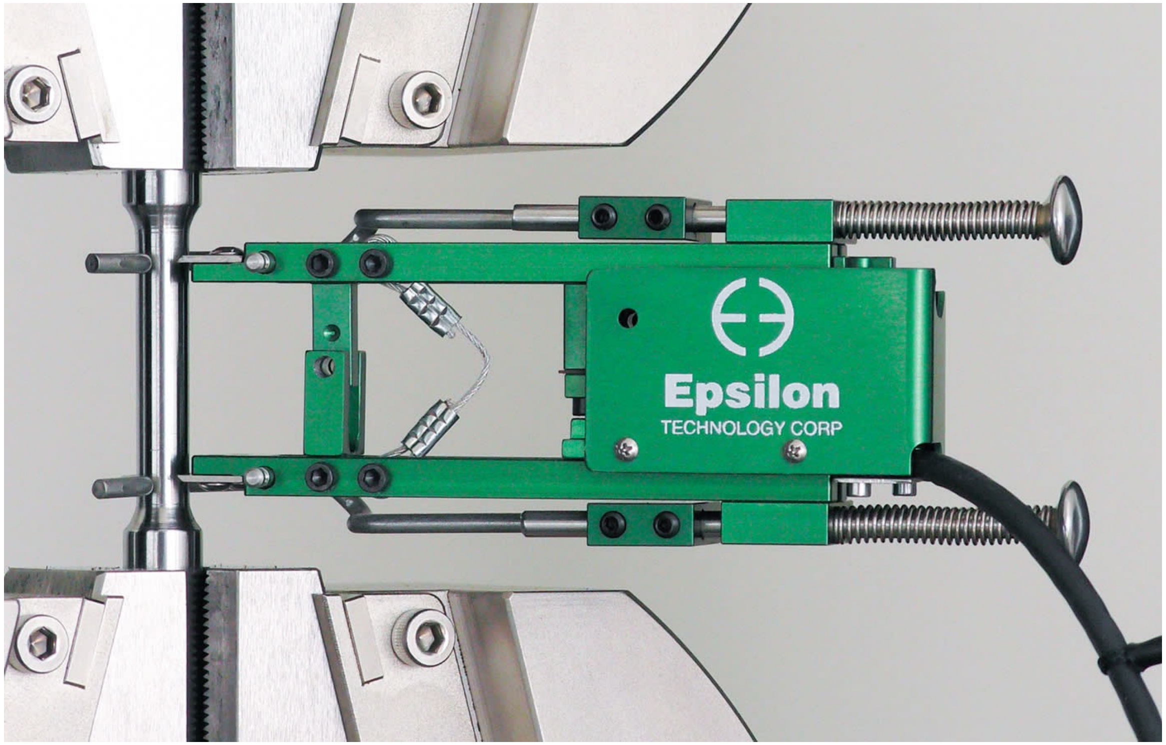
$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.25 \\ 0 & 0.25 & 0 \end{bmatrix}$$

응력과 변형률의 실험 측정법

- 응력과 변형률을 측정하는 방법에 대해 간단히 의논해보자.
- Q. 일축 인장(uniaxial tension test; uniaxial tensile test)법에 대해 배웠는가?
- Q. 일축 인장에서는 어떤 기계적 성질들을 취득할 수 있는가?

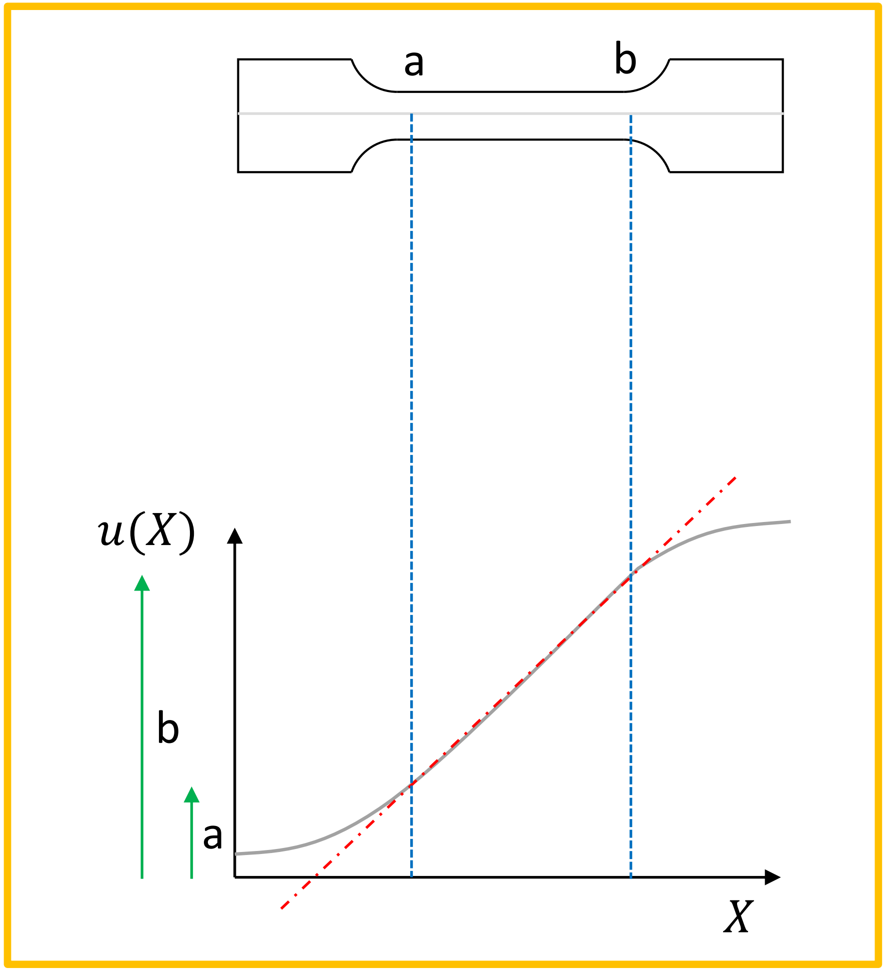
Measuring strains

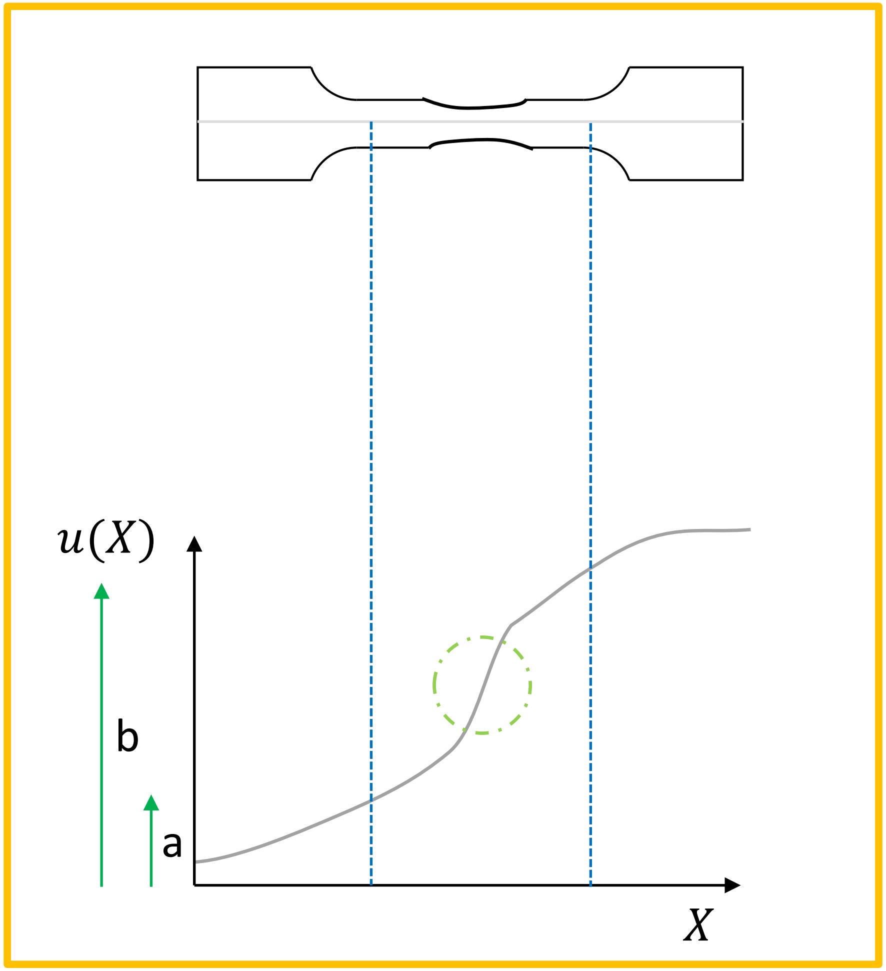
Measuring Relative motion of two points U(0), U(l)



two points 측정

- 일축인장 실험에서는 서로 '멀리' 떨어진 두 점을 활용하여 변형의 정도 (변형률)를 측정한다.





Deformation gradient

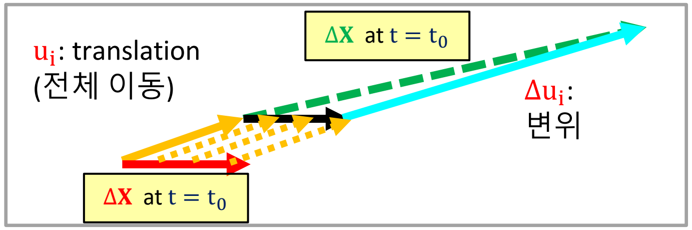
변위 구배($\nabla \boldsymbol{u}, u_{ij} = \frac{\partial u_i}{\partial X_j}$) 와 더불어
중요한 물리량 중 하나는
Deformation gradient tensor *F*:

$$F_{ij} = \frac{\partial u_i}{\partial X_j} + \delta_{ij}$$

δ_{ij} 는 Kronecker delta 라 불리며 다음의 성질을 따른다.

If $i = j, \delta_{ij} = 1$

If $i \neq j, \delta_{ij} = 0$



F 의 중요 성질:

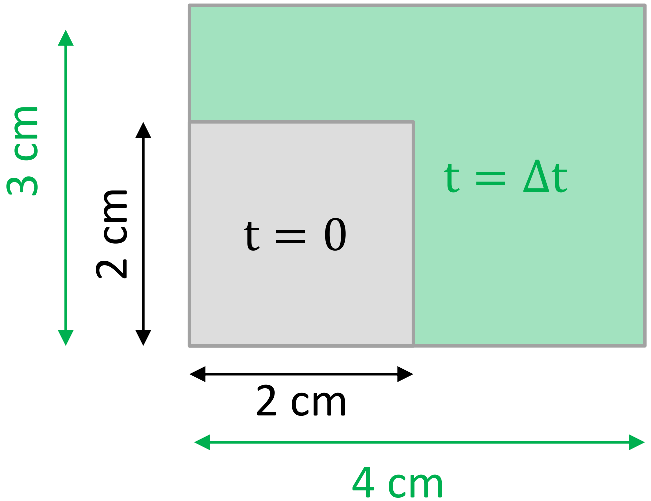
$$\Delta x_i^{t=t_1} = F_{ij} \Delta X_j^{t=t_0}$$

Linearly mapping $\Delta X_j^{t=t_0}$ to $\Delta x_i^{t=t_1}$ (*see Chapter Vectors and Matrices)

예제

한 물체가 다음과 같이 어떠한 운동에 의해 ‘**균일하게**’ 변형이 되었다.

한 점의 좌표: (X_1, X_2)



$$u_1 = 1X_1$$
$$u_2 = 0.5X_2$$

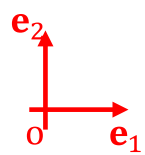
Displacement gradient tensor

$$u_{11} = \frac{\partial u_1}{\partial X_1} = \frac{4-2}{2}$$

$$u_{22} = \frac{\partial u_2}{\partial X_2} = \frac{3-2}{2}$$

$$u_{21} = \frac{\partial u_2}{\partial X_1} = 0$$

$$u_{12} = \frac{\partial u_1}{\partial X_2} = 0$$



2차원 좌표계
($\mathbf{e}_1, \mathbf{e}_2$ basis vectors)

$$u_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$F_{ij} = \frac{\partial u_i}{\partial X_j} + \delta_{ij}$$

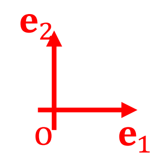
$$= \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 1.5 \end{bmatrix}$$

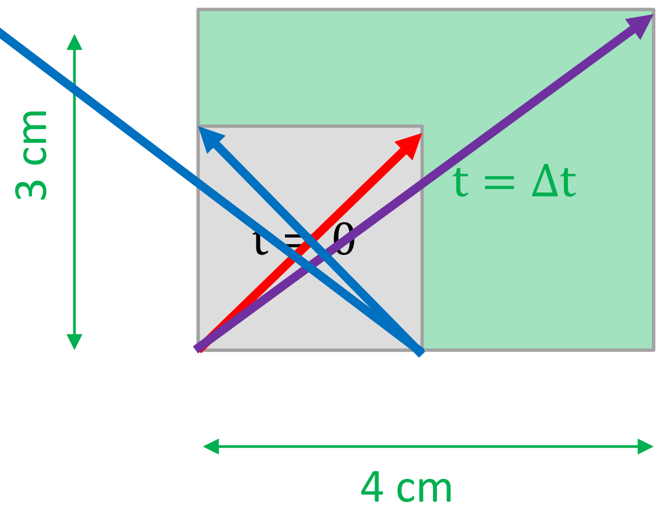
예제

한 물체가 다음과 같이 어떠한 운동에 의해 ‘**균일하게**’ 변형이 되었다.

한 점의 좌표: (X_1, X_2)



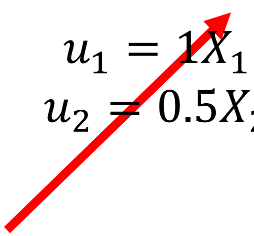
2차원 좌표계
($\mathbf{e}_1, \mathbf{e}_2$ basis vectors)



$$\frac{\partial u_i}{\partial X_j} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

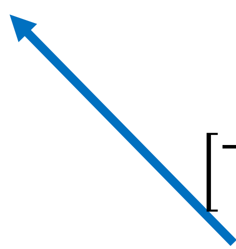
$$\mathrm{F_{ij}} = \begin{bmatrix} 2 & 0 \\ 0 & 1.5 \end{bmatrix}$$

$$u_1 = 1X_1$$
$$u_2 = 0.5X_2$$



$$\boldsymbol{l} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\mathrm{F_{ij}l_j} = \begin{bmatrix} 2 & 0 \\ 0 & 1.5 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = [4 \quad 3]$$



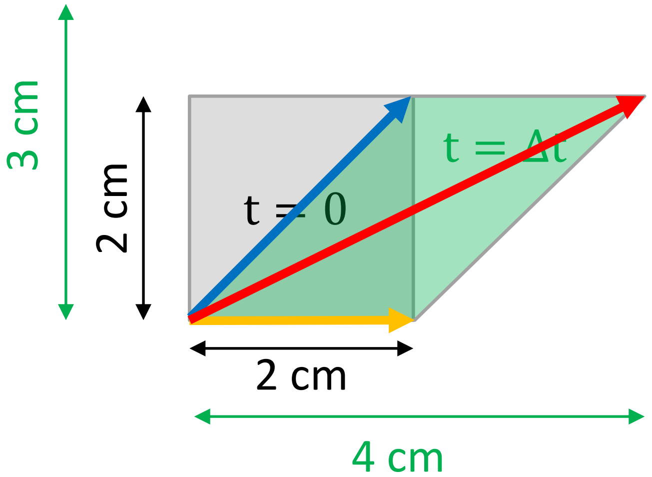
$$\begin{bmatrix} -2 \\ 2 \end{bmatrix} \qquad \mathrm{F_{ij}l_j} = \begin{bmatrix} 2 & 0 \\ 0 & 1.5 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \end{bmatrix} = [-4 \quad 3]$$

$\mathrm{F_{ij}}$ does not account for ‘translation’.
Translation is ‘**lost**’ when take derivatives of $\boldsymbol{u}$

예제

$$u_1 = X_2$$
$$u_2 = 0$$

$$F_{ij} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}$$



$[2,0] \xrightarrow{} \; F_{ij} \; \xrightarrow{} [2,0]$

$$\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \rightarrow \quad 2 = 2F_{11}, 0 = 2F_{21}$$

$$\rightarrow F_{11} = 1, F_{21} = 0$$

$$F_{ij} = \begin{bmatrix} 1 & F_{12} \\ 0 & F_{22} \end{bmatrix}$$

$[2,2] \xrightarrow{} \; F_{ij} \; \xrightarrow{} [4,2]$

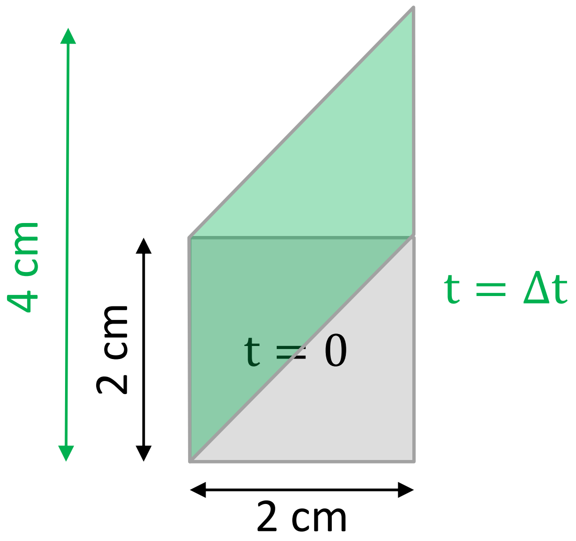
$$\begin{bmatrix} 1 & F_{12} \\ 0 & F_{22} \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$2 + 2F_{12} = 4$$
$$2F_{22} = 2$$

$$F_{ij} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

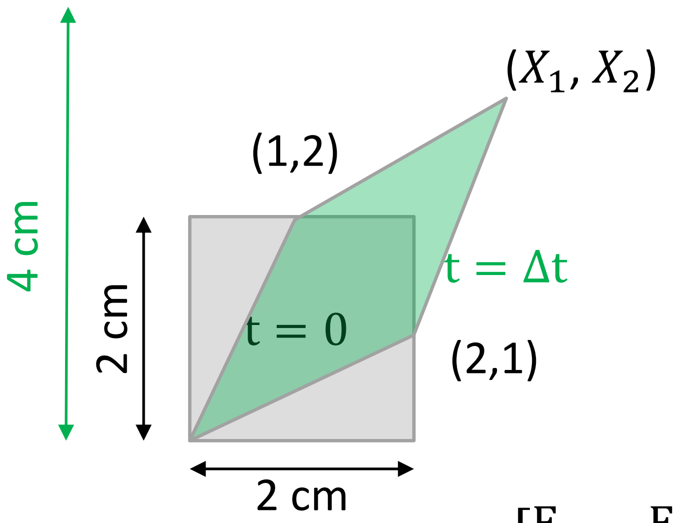
Q. Can you find u_{ij} for this deformation?

예제



Find out the deformation gradient for the given deformation illustrated in the left figure.

예제



Find out the deformation gradient for the given deformation illustrated in the left figure.

$$\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightarrow \begin{matrix} 2 = 2F_{11} \\ 1 = 2F_{21} \end{matrix} \qquad \begin{matrix} F_{11} = 1 \\ F_{21} = 0.5 \end{matrix}$$

$$u_1 = 0.5X_2$$
$$u_2 = 0.5X_1$$

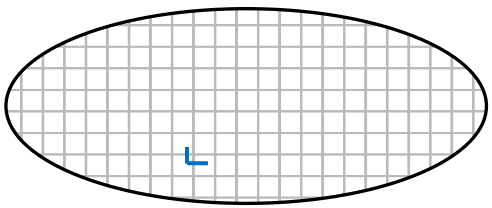
$$\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightarrow \begin{matrix} 2 = 2F_{22} \\ 1 = 2F_{12} \end{matrix} \qquad \begin{matrix} F_{22} = 1 \\ F_{12} = 0.5 \end{matrix}$$

$$\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Displacement and strain

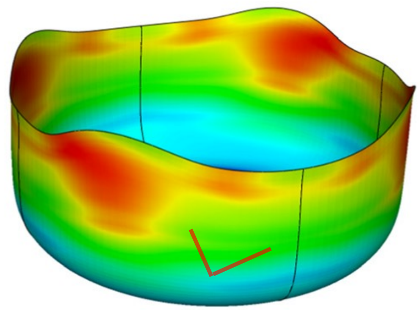
- Goal: Displacement와 strain의 관계를 이해하고 더 나아가 displacement에서 strain을 '추출' 해낼 수 있는 방법을 이해한다.

Blank sheet



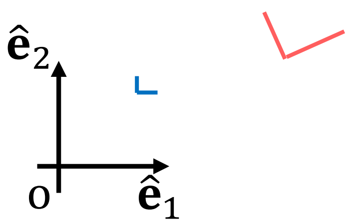
변형 →

Cup

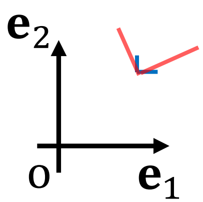


- Translation
- Rotation
- Extension (Contraction) and shear

1. 공통 좌표계에서 표기



2. Translation 제거



→ Displacement gradient tensor ($\nabla \mathbf{u}$, $\mathrm{u_{ij}} = \frac{\partial u_i}{\partial x_j}$)

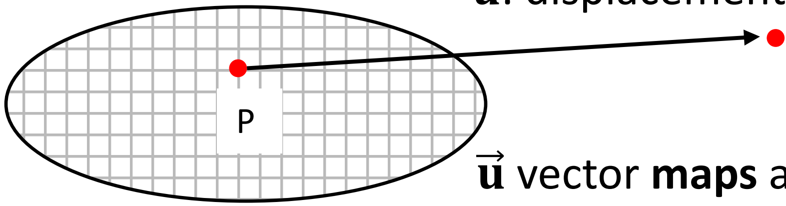
3. Rotation 제거

→ Strain = The 'symmetric' part of displacement gradient tensor

Displacement and strain

Displacement: 특정 한 점이 차지하던 position을 또 다른 position으로 옮겨준다.

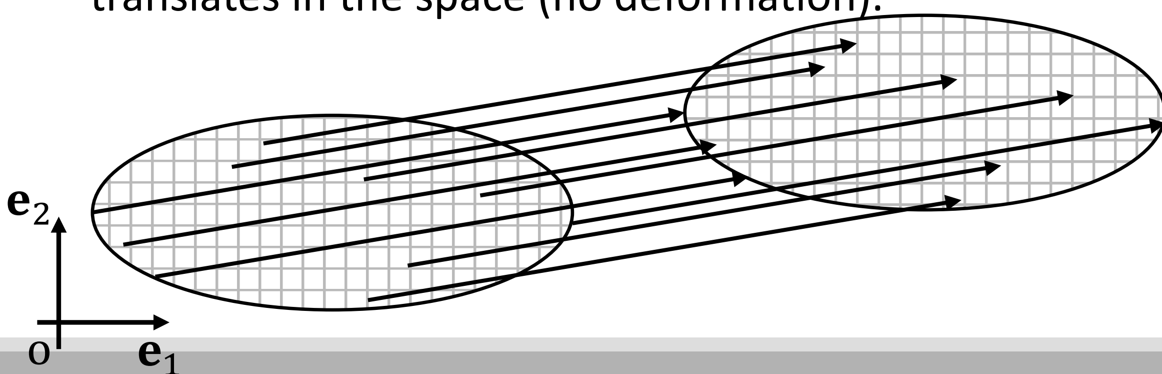
$\vec{\mathbf{u}}$: displacement vector



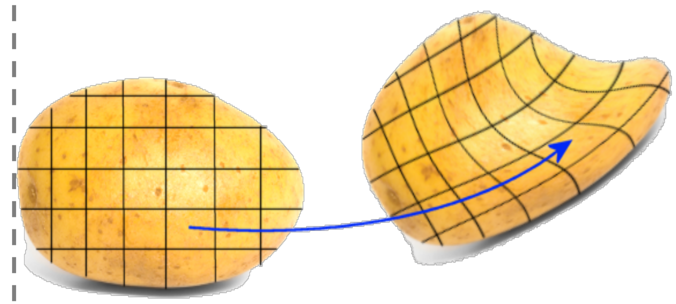
$\vec{\mathbf{u}}$ vector **maps** a single point P to P`

$\vec{\mathbf{u}}(x_1, x_2)$: displacement vector **field** maps various points to various points.

In case $\vec{\mathbf{u}}$ field is uniform, which means that $\vec{\mathbf{u}}$ is the same for all points, the material only translates in the space (no deformation).



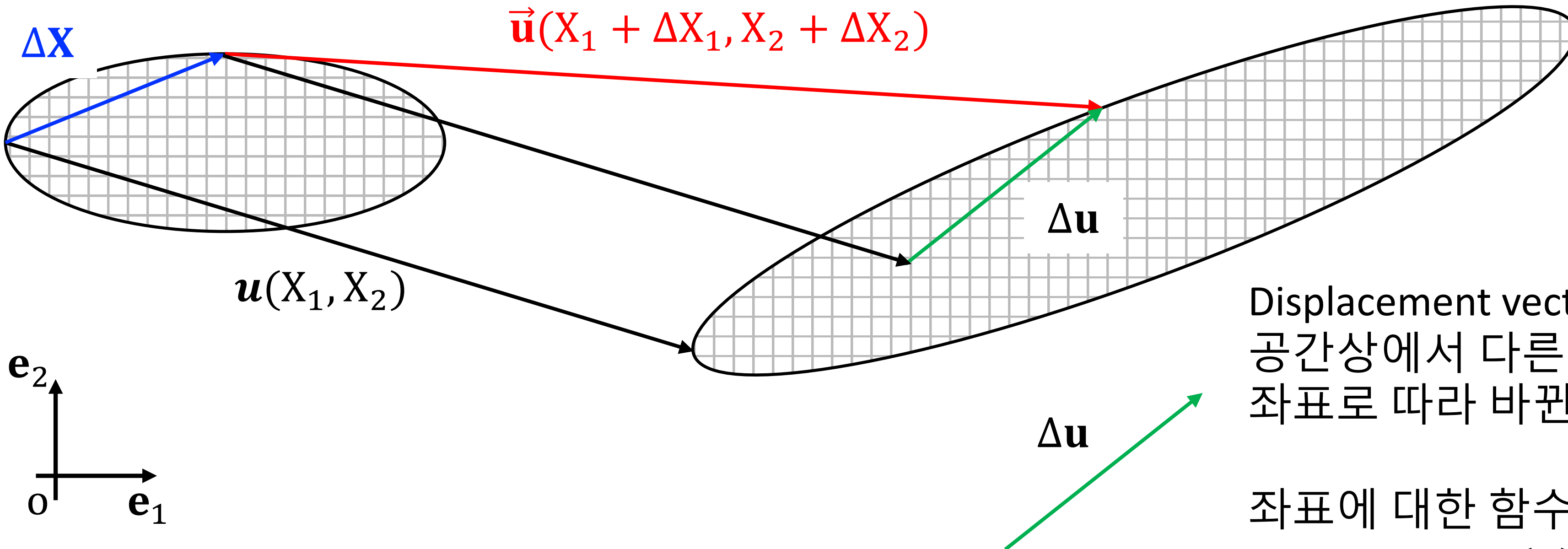
Deformation occurs only when $\vec{\mathbf{u}}$ field is not uniform, which means that $\vec{\mathbf{u}}$ varies when changing the locations.



Warning: there are cases that $\vec{\mathbf{u}}$ field is not uniform, but no deformation occurs (We'll get back to this later).

Displacement and strain

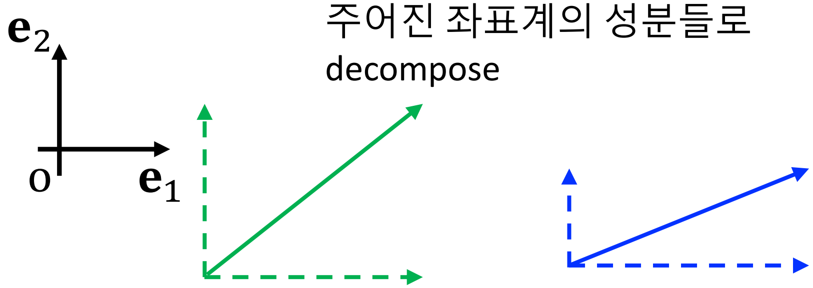
In case $\boldsymbol{u}(\mathrm{x}_1, \mathrm{x}_2)$ is not uniform (case 1)



Displacement vector 가
공간상에서 다른
좌표로 따라 바뀐 차이

좌표에 대한 함수, 즉
$$\Delta\mathbf{u} = \Delta\mathbf{u}(\mathbf{X})$$

파란 화살표로 옮겨진 점의 물질은 기준이 되는 점에 비해 녹색으로 표현된 만큼 차이나는 점으로 옮겨졌다.



$\Delta\mathbf{u} = \Delta\mathrm{u}_1\mathbf{e}_1 + \Delta\mathrm{u}_2\mathbf{e}_2$

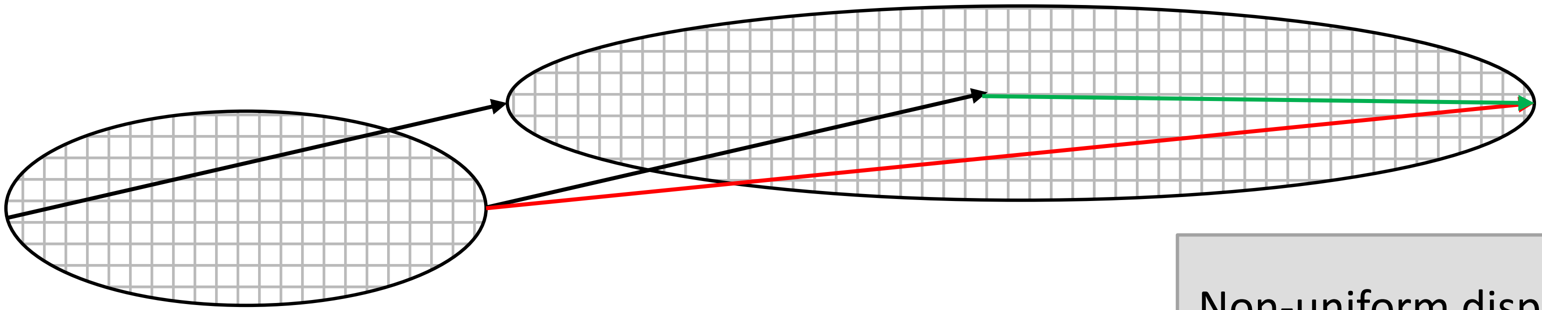
$\Delta\mathbf{x} = \Delta\mathrm{x}_1\mathbf{e}_1 + \Delta\mathrm{x}_2\mathbf{e}_2$

$\vec{\mathbf{u}}$ 가 공간에 따라 어떻게 얼마나 달라지는지 나타내는 수학적 방법 (gradient)

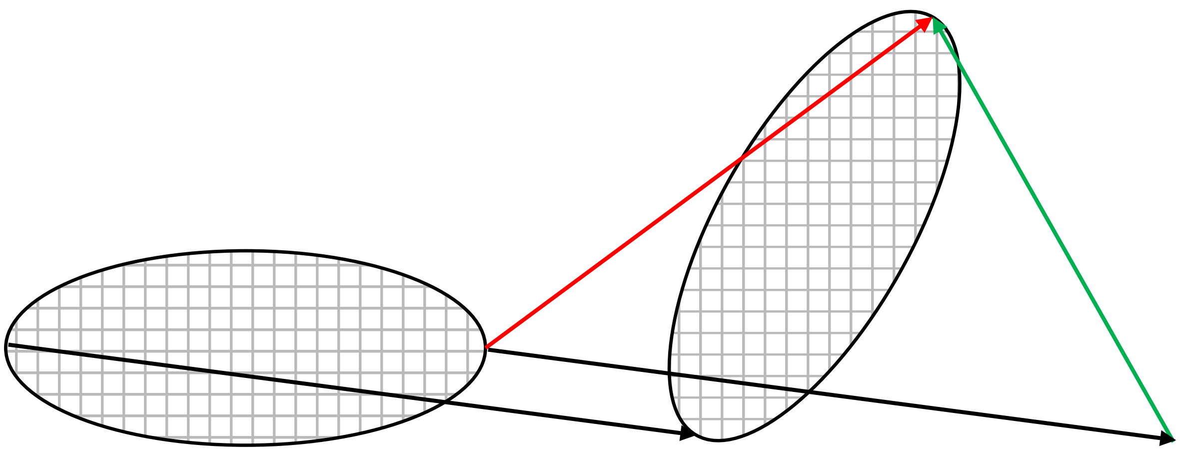
$\frac{\Delta\mathbf{u}(\mathbf{X})}{\Delta\mathbf{X}}$ 로 표기 하자.

Displacement and strain

In case $\vec{\mathbf{u}}(X_1, X_2, X_3)$ is not uniform (case 1; pure stretching)



In case $\vec{\mathbf{u}}(X_1, X_2, X_3)$ is not uniform (case 2; pure rotation)



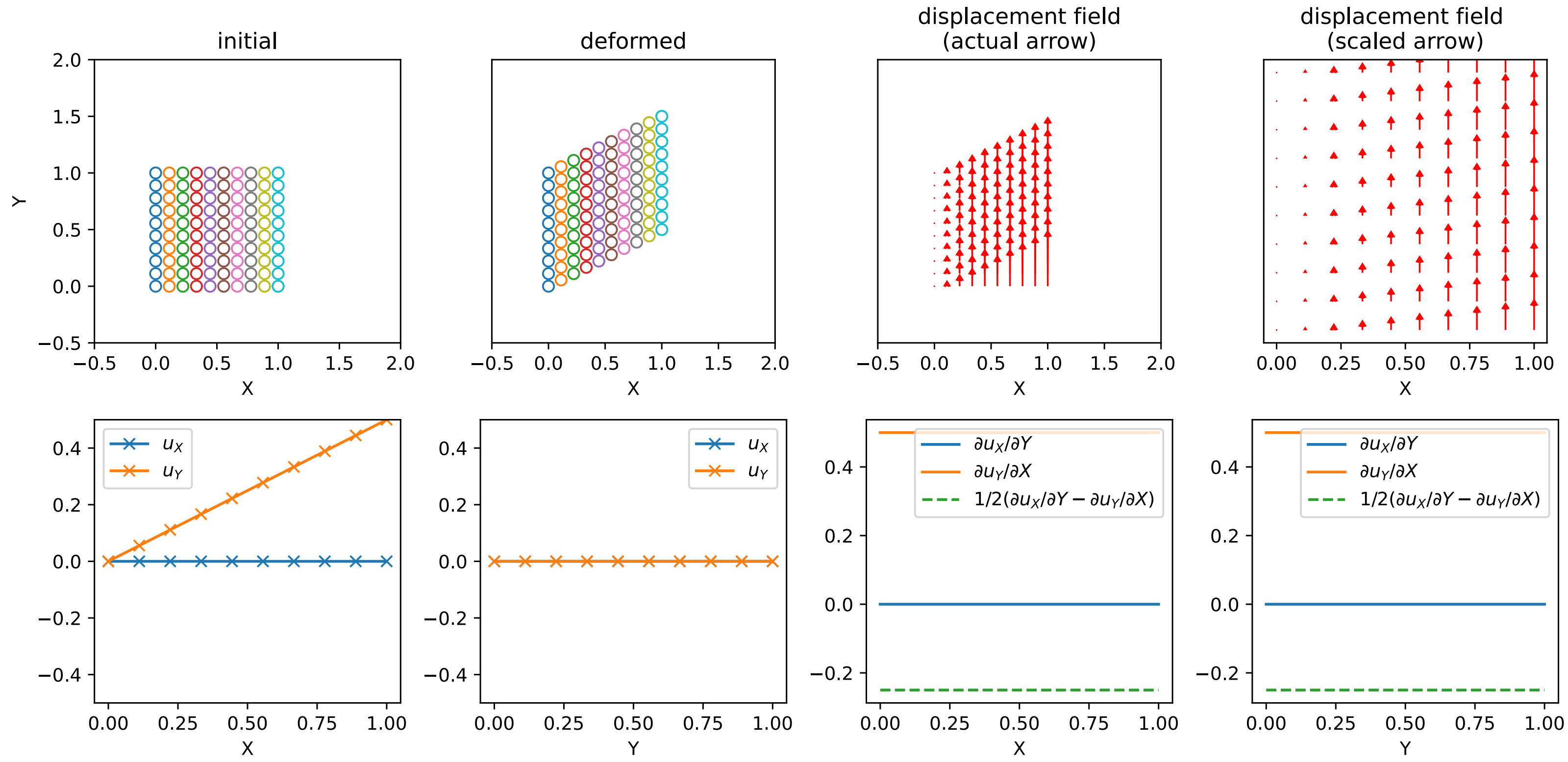
Non-uniform displacement field does not always mean that the material is 'deformed'.
Non-uniform displacement field may contain a contribution from 'rotation'.

Therefore, if you want to 'extract' only the 'deformation', you have to exclude 'rotational' contribution from the displacement field.

Shear components of $\nabla u_{ij}, d_{ij}, \frac{\partial u_i}{\partial X_j}$

$$\frac{\partial u_i}{\partial x_j} = \varepsilon_{ij} + \omega_{ij}, \qquad \varepsilon_{\mathrm{ij}} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

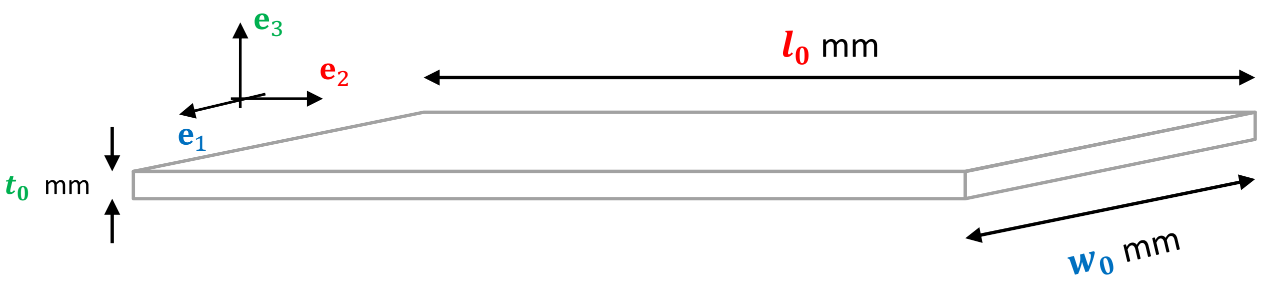
$$\omega_{\mathrm{ij}} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \varepsilon_{ij}$$



Displacement gradient to strain

- *Small strain theory*
  - $$\frac{\partial u_i}{\partial x_j} = \lim_{\Delta X_j \to 0} \frac{\Delta u_i}{\Delta X_j} = \frac{\partial u_i}{\partial X_j}$$
  - $$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$
- 위 특성으로 인해
  - $$\varepsilon_{ij}^T = \varepsilon_{ij}$$

Example



위의 금속 판재에 냉간 압연을 하여 두께,너비,길이가 각각 t_1, w_1, l_1 으로 바뀌었다.

- 부피 변형률 $\ln\left(\frac{V_1}{V_0}\right)$ 값을 $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}$ 요소로 표현하여라.

$$\frac{V_1}{V_0} = \frac{l_1 w_1 t_1}{l_0 w_0 t_0} \qquad (1)$$

(1) 의 양변에 자연 로그를 사용하면

$$\ln\left(\frac{V_1}{V_0}\right) = \ln\left(\frac{l_1}{l_0}\right) + \ln\left(\frac{w_1}{w_0}\right) + \ln\left(\frac{t_1}{t_0}\right) = \varepsilon_{22} + \varepsilon_{11} + \varepsilon_{33}$$

따라서 부피변화가 없다면, 즉 $\ln(1) = 0$, 따라서 $\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = 0$

Example

- 전단 변형률은 부피 변화와 무관하다.

- Let's check

output										
		This excell sheet proves a means of coordinate system transformation								
			angle	radian						
		phi1	45	0.785						
Three Euler angles		Phi	30	0.524						
		phi2	20	0.349						
					transformation matrix R			(transformation matrix)^t= R^t=R^-1		
삼각 함수 값들										
cos(phi1)	0.707	sin(phi1)	0.707		0.455	0.874	0.171	0.455	-0.817	0.354
cos(Phi)	0.866	sin(Phi)	0.500		-0.817	0.334	0.470	0.874	0.334	-0.354
cos(phi2)	0.940	sin(phi2)	0.342		0.354	-0.354	0.866	0.171	0.470	0.866
2nd rank tensor in matrix form					R.T			R^t.R.T	**2nd rank tensor after coordinate transformation**	
1	0	0			0.455	-0.874	0.342	-0.498	-0.503	0.766
0	-1	0			-0.817	-0.334	0.940	-0.503	0.998	0.643
0	0	2			0.354	0.354	1.732	0.766	0.643	1.500
1st rank tensor (i.e., vector) in array form					R.v 1st rank tensor (vector) after coordinate transformation					
1					0.4550193	-0.8172866	0.3535534			
0										
0										

Finite strain theory

- $\boldsymbol{F} = \boldsymbol{R} \cdot \boldsymbol{U} = \boldsymbol{V} \cdot \boldsymbol{R}$

- **U** (or **V**) represents ‘stretching’ and **R** represents ‘rotation’.

- There are several strain measures.
- However, stress is usually quantified using Cauchy stress tensor.

References and acknowledgements

- References
  - An introduction to Continuum Mechanics – M. E. Gurtin
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  - Fundamentals of metal forming (R. H. Wagoner, J-L Chenot)
  - http://www.continuummechanics.org (very good on-line reference)
  - http://youngung.github.io/tensors

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  - Some images presented in this lecture materials were collected from Wikipedia.