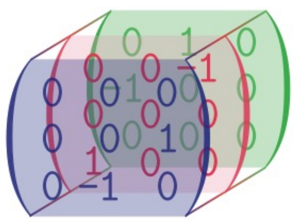
Cross product and permutation symbol

크기가 1이고 수직관계(orthonormal)인 두 basis 벡터 사이의 cross product의 정의:

$$\mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijk}\mathbf{e}_k \ (i, j\text{: free; } k\text{: dummy})$$

The symbol ϵ_{ijk} is called the alternating symbol (or more commonly permutation symbol and more formally Levi-Civita symbol).

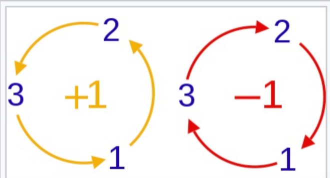
$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } i,j,k \text{ are in cyclic order and not repeated } (123, 231, 312), \\ -1, & \text{if } i,j,k \text{ are not in cyclic order and not repeated} (132, 213, 321), \\ 0, & \boxed{\text{if any of } i,j,k \text{ are repeated.}} \end{cases}$$

$\epsilon_{\text{ijk}} =$ 

where i is the depth (blue: $i = 1$; red: $i = 2$; green: $i = 3$), j is the row and k is the column.

If $\boldsymbol{a}\times\boldsymbol{b} = \boldsymbol{c}$, then

$$c_k = \boldsymbol{c}\cdot\mathbf{e}_k = a_i b_j \mathbf{e}_i \times \mathbf{e}_j \cdot \mathbf{e}_k$$



For the indices (i, j, k) in ϵ_{ijk} the values 1, 2, 3 occurring in the [orange] cyclic order (1, 2, 3) correspond to $\epsilon = +1$, while occurring in the [red] reverse cyclic order correspond to $\epsilon = -1$, otherwise $\epsilon = 0$.

$$\boldsymbol{a}\times\boldsymbol{b} = (a_i\mathbf{e}_i)\times(b_j\mathbf{e}_j) = a_i b_j(\mathbf{e}_i\times\mathbf{e}_j) = a_i b_j \epsilon_{ijk}\mathbf{e}_k$$

Ex) Prove $\epsilon_{ijk}\epsilon_{ijk} = 6$

$$\epsilon_{ijk}\epsilon_{ijk} = \sum_i \sum_j \sum_k \epsilon_{ijk}^2 = \epsilon_{1mn}^2 + \epsilon_{2op}^2 + \epsilon_{3qr}^2$$

https://en.wikipedia.org/wiki/Levi-Civita_symbol

예제

Cross product between two orthonormal basis vectors: $\mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijk}\mathbf{e}_k$ (i, j: free; k: dummy)

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } i,j,k \text{ are in cyclic order and not repeated } (123, 231, 312), \\ -1, & \text{if } i,j,k \text{ are not in cyclic order and not repeated} (132, 213, 321), \\ 0, & \boxed{\text{if any of } i,j,k \text{ are repeated.}} \end{cases}$$

Q1) 벡터 $\boldsymbol{a}$와 $\boldsymbol{b}$가 각각 (1,2,3) 그리고 (1,0,1) 일 때, $\boldsymbol{a} \times \boldsymbol{b}$를 구하시오.

$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$ 라하면,

$\boldsymbol{c} = c_i \mathbf{e}_i$이므로 (벡터 분해)

$$c_k \mathbf{e}_k = a_i b_j \epsilon_{ijk} \mathbf{e}_k$$

Only one free index: k. 따라서, $k = 1,2,3$인 경우를 따지면…

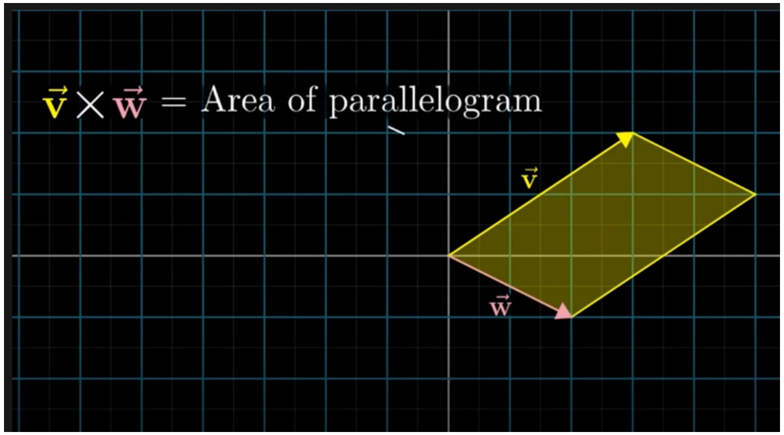
$$c_1 \mathbf{e}_1 = a_i b_j \epsilon_{ij1} \mathbf{e}_1 = \sum_i^3 \sum_j^3 a_i b_j \epsilon_{ij1} \mathbf{e}_1 = a_2 b_3 \mathbf{e}_1 - a_3 b_2 \mathbf{e}_1$$

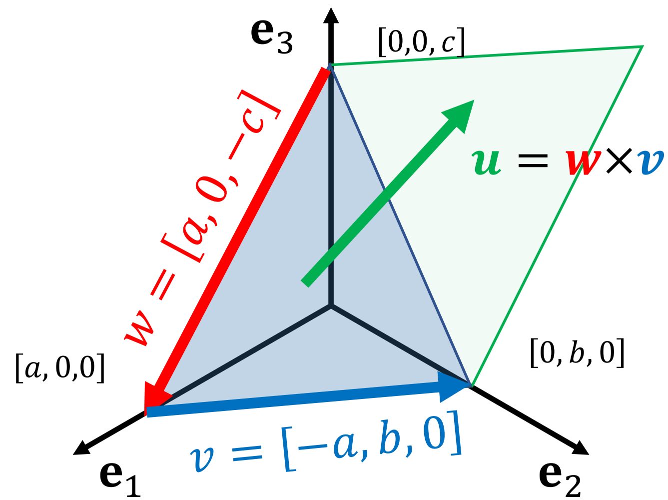
$$c_2 \mathbf{e}_2 = a_i b_j \epsilon_{ij2} \mathbf{e}_2 = \sum_i^3 \sum_j^3 a_i b_j \epsilon_{ij2} \mathbf{e}_2 = -a_1 b_3 \mathbf{e}_2 + a_3 b_1 \mathbf{e}_2$$

$$c_3 \mathbf{e}_3 = a_i b_j \epsilon_{ij3} \mathbf{e}_3 = \sum_i^3 \sum_j^3 a_i b_j \epsilon_{ij3} \mathbf{e}_3 = a_1 b_2 \mathbf{e}_3 - a_2 b_1 \mathbf{e}_3$$

Area of inclined triangle calculated by using cross-product

REF: https://youtu.be/eu6i7WJeinw





The area of triangle: $\frac{|\boldsymbol{u}|}{2} = \frac{|\boldsymbol{w}\times\boldsymbol{v}|}{2}$

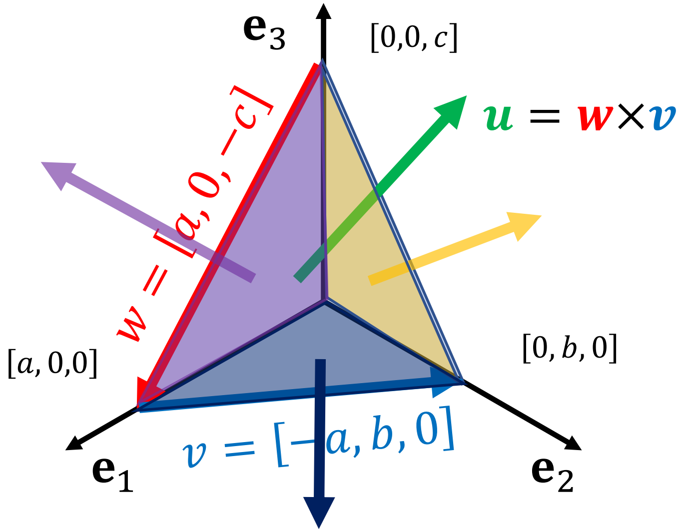
The unit normal vector of the traingle: $\frac{\boldsymbol{u}}{|\boldsymbol{u}|}$

$$\frac{|\boldsymbol{u}|}{2} = \frac{\sqrt{(bc)^2 + (ac)^2 + (ab)^2}}{2}$$

$$\frac{\boldsymbol{u}}{|\boldsymbol{u}|} = \frac{[bc, ac, ab]}{\sqrt{(bc)^2 + (ac)^2 + (ab)^2}}$$

$$\boldsymbol{w}\times\boldsymbol{v} = \boldsymbol{u}$$
$$(w_i\mathbf{e}_i)\times(v_j\mathbf{e}_j) = w_i v_j \epsilon_{ijk}\mathbf{e}_k = u_k\mathbf{e}_k$$
$$u_1 = w_i v_j \epsilon_{ij1} = w_2 v_3 - w_3 v_2 = -(-c)b = bc$$
$$u_2 = w_i v_j \epsilon_{ij2} = w_3 v_1 - w_1 v_3 = (-c)(-a) = ac$$
$$u_3 = w_i v_j \epsilon_{ij3} = w_1 v_2 - w_2 v_1 = ab$$

Relations between triangular surfaces (will be useful for Cauchy tetrahedron)



$$\boldsymbol{r} = [0,0,c]\times[0,b,0] = -cb\mathbf{e}_1$$

$$\boldsymbol{q} = [a,0,0]\times[0,0,c] = -ac\mathbf{e}_2$$

$$\boldsymbol{s} = [0,b,0]\times[a,0,0] = -ba\mathbf{e}_3$$

Volume of tetrahedron: $\frac{abc}{6}$

Area: $\frac{|\boldsymbol{u}|}{2} = \frac{\sqrt{(bc)^2+(ac)^2+(ab)^2}}{2}$

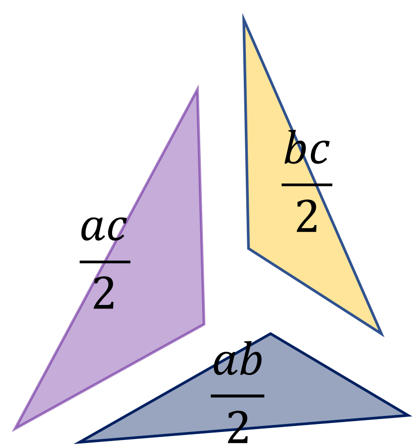
Unit normal vector: $\frac{\boldsymbol{u}}{|\boldsymbol{u}|} = \frac{[bc,ac,ab]}{\sqrt{(bc)^2+(ac)^2+(ab)^2}} = [bc, ac, ab]\left(\frac{1}{2\times\text{Area}}\right)$

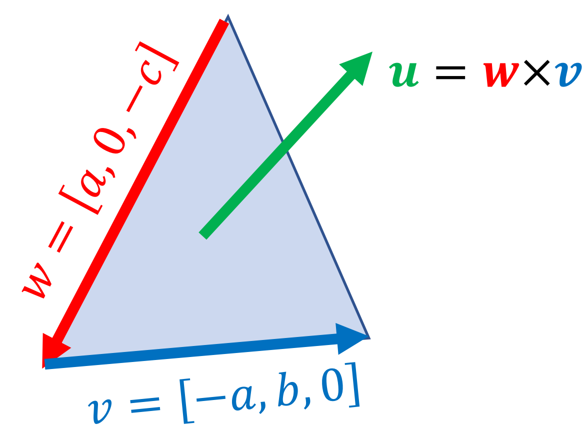
Confirm

$$\frac{u_1}{|\boldsymbol{u}|}\frac{\sqrt{(bc)^2+(ac)^2+(ab)^2}}{2} = \frac{bc}{2}$$

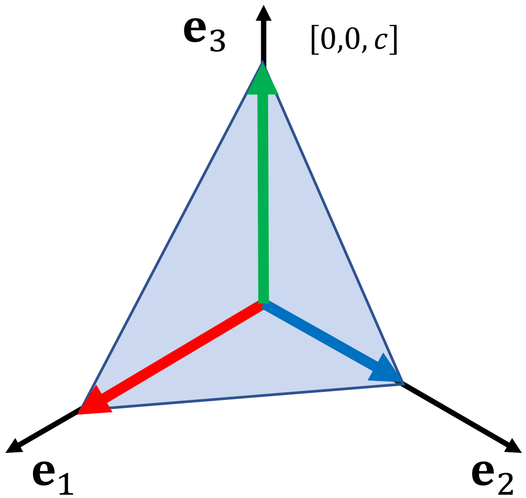
$$\frac{u_2}{|\boldsymbol{u}|}\frac{\sqrt{(bc)^2+(ac)^2+(ab)^2}}{2} = \frac{ac}{2}$$

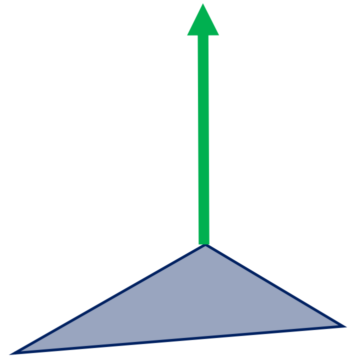
$$\frac{u_2}{|\boldsymbol{u}|}\frac{\sqrt{(bc)^2+(ac)^2+(ab)^2}}{2} = \frac{ab}{2}$$

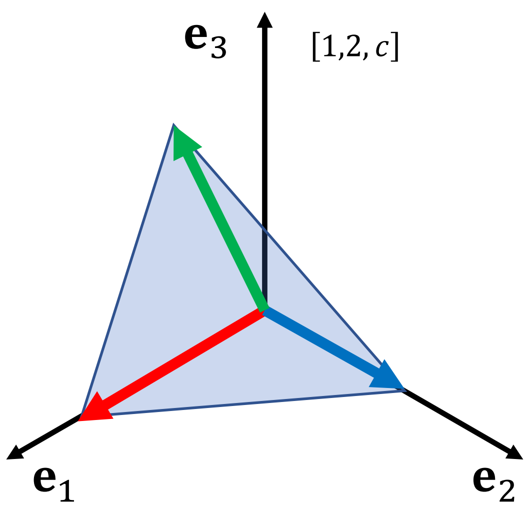


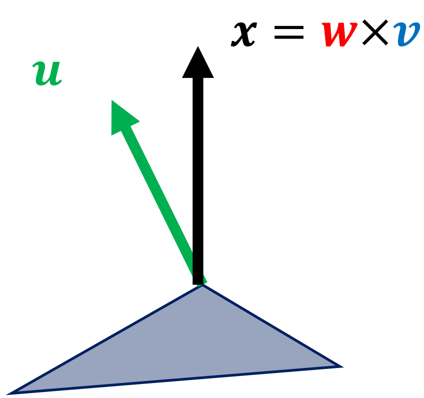


Relations between triangular surfaces (will be useful for Cauchy tetrahedron)

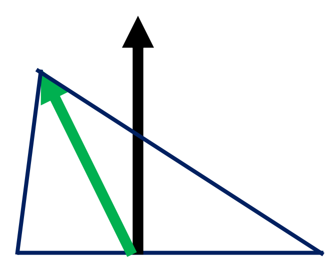








$$V = \frac{(\boldsymbol{w} \times \boldsymbol{v}) \cdot \boldsymbol{u}}{6}$$



Integration (scalars)

a is a quantity that is varying with respect to time t. If you know the initial value of it, (i.e., $a(t=0)$ and you know $\frac{da}{dt}$ in all time stamps, you'll be able to calculate $a(t=\tau)$ via:

$a(t=\tau) = a(t=0) + \int_0^\tau \frac{da}{dt}\, dt$, this is sometimes written in short:

$$a_\tau = a_{(0)} + \int_0^\tau \frac{da}{dt}\, dt$$

You could have an analytic expression of the above, (or not). In case of former, you'd have Something like

$$a(\tau) = a_0 + \tau^2 + \cos(\tau)\exp(\tau) \ldots$$

In case you cannot obtain an analytic expression, you can could 'numerically' obtain the solution.

Integration (vectors, tensors)

$\boldsymbol{a}$ is a quantity that is varying with respect to time t. If you know the initial value of it, (i.e., $\boldsymbol{a}(t=0)$ and you know $\frac{da_i}{dt}$ (i being the free index) in all time stamps, you'll be able to calculate $\boldsymbol{a}(t=\tau)$ via:

$\boldsymbol{a}(t=\tau) = \boldsymbol{a}(t=0) + \int_0^\tau \frac{d\boldsymbol{a}}{dt}\, dt$, this is sometimes written in short:

$$\boldsymbol{a}_\tau = \boldsymbol{a}_{(0)} + \int_0^\tau \frac{d\boldsymbol{a}}{dt}\, dt$$

You could have an analytic expression of the above, (or not). In case of former, you'd have Something like

$$\boldsymbol{a}(\tau) = \boldsymbol{a}_0 + \cos(\tau)\exp(\tau)\,\mathbb{M}\!:\boldsymbol{b}\ \ldots$$

In case you cannot obtain an analytic expression, you can could 'numerically' obtain the solution.

Example

질량이 10kg 인 한 포탄이 최초 발사각 30°를 보이며 날아가는 직선방향에서 10 m/sec의 속도의 초기 속도를 가진다. 공기의 저항을 무시하고, 중력가속도를 10 m/s^2이라 가정한다면, 포탄10 초 이후의 위치는 어디인가?

$$x(\tau) = x(\tau = 0) + \int_0^\tau \frac{dx}{dt}\, dt$$

$$\boldsymbol{x}(\tau) = \boldsymbol{x}(\tau = 0) + \int_0^\tau \frac{d\boldsymbol{x}}{dt}\, dt \quad = \int_0^\tau \boldsymbol{v}(t)\, dt$$

$$\boldsymbol{v}(\tau) = \boldsymbol{v}(\tau = 0) + \int_0^\tau \frac{d\boldsymbol{v}}{dt}\, dt$$

$$\boldsymbol{v}(\tau) = \boldsymbol{v}(\tau = 0) + \int_0^\tau \boldsymbol{a}(t) dt$$

$$x_i(\tau) = x_i(\tau = 0) + \int_0^\tau \frac{dx_i}{dt}\, dt = \int_0^\tau v_i(t)\, dt$$

$$v_i(\tau) = v_i(\tau = 0) + \int_0^\tau \frac{dv_i}{dt}\, dt$$

$$v_i(\tau) = v_i(\tau = 0) + \int_0^\tau a_i(t) dt$$

Summary

- Nomenclature
- Why vectorial quantities are required for physical laws?
- Vector operations
  - ✓Basic vector operations: addition(+), subtraction(-), inner dot(·)
  - ✓Use the same coordinate system for vector operations
  - ✓Dyadic operation (⨂)and Schmid tensor
  - ✓Identity matrix (Kronecker delta)
  - ✓Transpose operation ($A_{ij}^{T} = A_{ji}$)
  - ✓Linear transformation (= linear map)
  - ✓Matrix operations: inner dot (·), double inner dot(:)
  - ✓Coordinate transformation
  - ✓Cross product (×) and its geometrical interpretation
- Time integration of vectorial quantities

$$x_i(\tau) = x_i(\tau = 0) + \int_0^{\tau} \frac{dx_i}{dt}\, dt$$

Reference

https://www.continuummechanics.org

https://lib.changwon.ac.kr/search/DetailView.ax?sid=1&cid=934390