Inverse transformation?

$$\mathbf{e}_i = \mathbf{e}_i 1 = \mathbf{e}_i(\mathbf{e}'_j \cdot \mathbf{e}'_j) = (\mathbf{e}_i \cdot \mathbf{e}'_j)(\mathbf{e}_j{}') = b_{ij}\mathbf{e}'_j$$

$$b_{kj}x'_j = b_{kj}a_{jl}x'_l = \delta_{kl}x'_l = x'_k$$

Earlier, we defined:

$$a_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j$$
$$b_{ij} = \mathbf{e}_i \cdot \mathbf{e}'_j$$

$$a_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j = \mathbf{e}_j \cdot \mathbf{e}'_i = b_{ji}$$

In summary we have:

$$\mathbf{x} = x'_i\mathbf{e}'_i = x_j\mathbf{e}_j$$
$$\mathbf{e}'_i = a_{ij}\mathbf{e}_j, \qquad \mathbf{e}_i = a_{ji}\mathbf{e}'_j$$
$$x'_i = a_{ij}x_j, \qquad x_i = a_{ji}x'_j$$
$$a_{ik}a_{jk} = a_{ki}a_{kj} = \delta_{ij}$$

If the inner dot product of a and b matrices:

$$a_{ik}b_{kj} = (\mathbf{e}'_i \cdot \mathbf{e}_k)(\mathbf{e}_k \cdot \mathbf{e}'_j)$$
$$= \mathbf{e}'_i \cdot (\mathbf{e}_k \cdot \mathbf{e}_k) \cdot \mathbf{e}'_j$$
$$= \mathbf{e}'_i \cdot \mathbf{e}'_j = \delta_{ij}$$

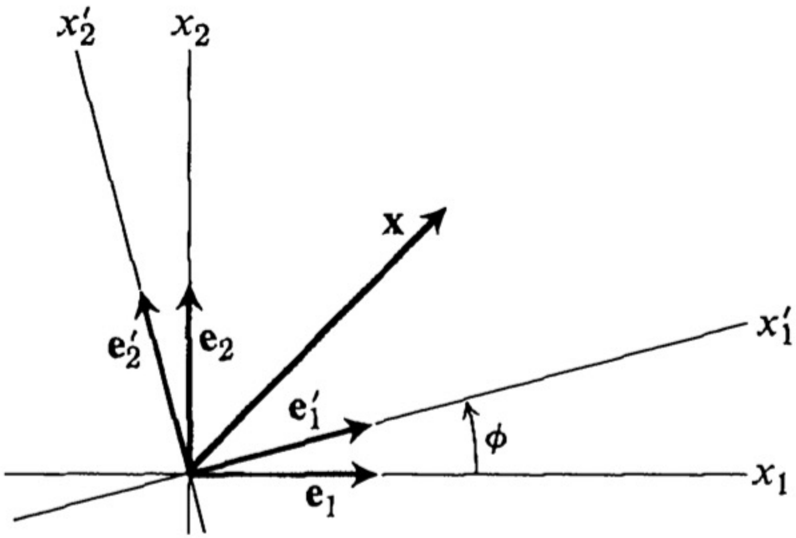
Scalar product is invariant under orthogonal transformations

$$\boldsymbol{x}' \cdot \boldsymbol{y}' = x_i' y_i' \quad = a_{ij} x_j a_{ik} y_k \quad = a_{ij} a_{ik} x_j y_k$$

$$= \delta_{jk} x_j y_k \; = x_j y_j \; = \boldsymbol{x} \cdot \boldsymbol{y}$$

$$a_{ij} a_{ik} = \left(a_{ji}\right)^T a_{ik} = b_{ji} a_{ik} = \delta_{jk}$$

Two dimensional case



https://en.wikipedia.org/wiki/List_of_trigonometric_identities

Shift by one quarter period
$\sin(\theta \pm \frac{\pi}{2}) = \pm \cos\theta$
$\cos(\theta \pm \frac{\pi}{2}) = \mp \sin\theta$

$$a_{ij} \equiv (\mathbf{e}_i' \cdot \mathbf{e}_j), \qquad \text{for } i, j = 1,2$$

$$[a_{ij}] = \begin{bmatrix} \cos\phi & \cos(90^\circ - \phi) \\ \cos(90^\circ + \phi) & \cos\phi \end{bmatrix}$$

Physical theories must be invariant to the choice of coordinate system

If we fix our attention on a physical vector (e.g. velocity) and then rotate the coordinate system $(K \rightarrow K')$, the vector will have different numerical components in the rotated coordinate system (as evident in the coordinate transformation rule we just discussed earlier). So we are led to realize that a vector is more than an ordered triple. Rather, it is many sets of ordered triples, which are related in a definite way. One still specifies a vector by giving three ordered numbers (components), but these three numbers are distinguished from an arbitrary collection of three numbers by including the law of coordinate transformation under rotation of the coordinate frame as part of the definition.

Thus, one physical vector may be represented by infinitely many sets of ordered triples. The particular triple depends on the chosen coordinate system of the observer.

This is important because physical laws (and results) must be the same regardless of coordinate system, that is, regardless of the orientation of observer's coordinate system.

Physical laws and coordinate system

- The importance of thinking of these quantities in terms of their transformation properties lies in the requirement that physical theories must be invariant under the change of the coordinate system.
- Physical laws should not be affected by the choice of a coordinate system.
- We’ll examine this using an example in what follows.

Newton's second law

Algebraic representation

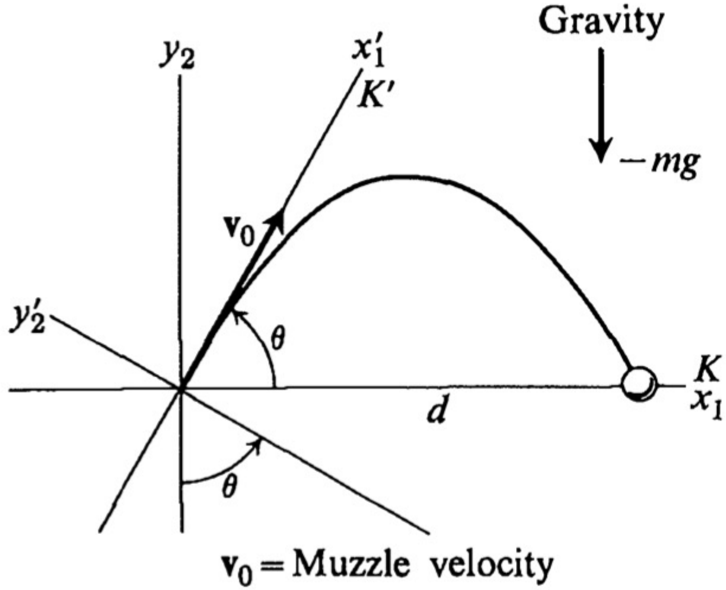
$$\boldsymbol{F} = m\boldsymbol{a} \rightarrow F_i = ma_i \rightarrow F_i = m\dot{v}_i$$

$$\dot{\boldsymbol{a}} = \frac{d\boldsymbol{a}}{dt}$$

$$F_i = m\dot{v}_i = m\ddot{x}_i$$

$$v_i = \frac{dx_i}{dt} = \dot{x}_i$$

$$a_i = \dot{v}_i = \frac{dv_i}{dt} = \frac{d\dot{x}_i}{dt} = \ddot{x}_i$$



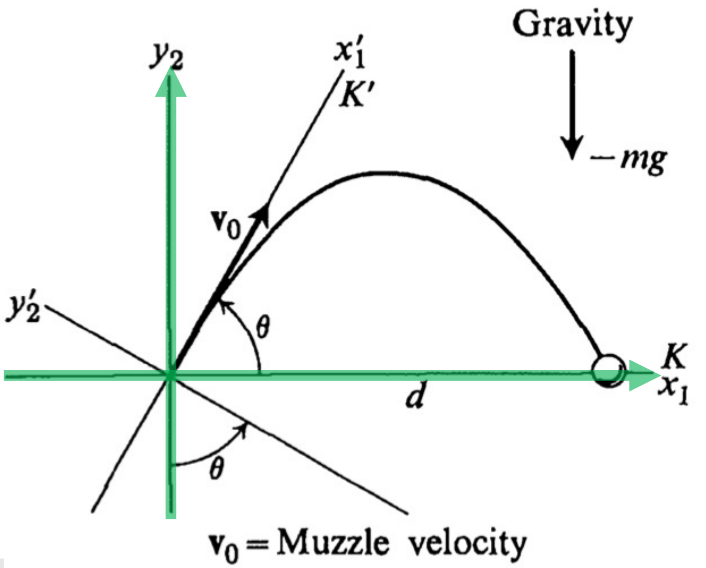
Let's assume acceleration $\ddot{x}_i$ is function of time, so that

$$\ddot{\boldsymbol{x}} \equiv \ddot{x}_i(t)$$

Furthermore, if we assume the mass is constant (which is quite usual), the second law is equation with the location $\boldsymbol{x}_i$ and its derivatives as variable – do not forget another variable time (t).

Newton's second law

$$F_i(t) = m\,\ddot{x}_i(t)$$



Let's use K coordinate system

1. Initial condition in terms of location (x_i) and velocity ($\dot{x}_i$):

$$x_i(0) = 0, \qquad \text{with } i = 1,2$$
$$\dot{x}_1(0) = v_0 \cos\theta$$
$$\dot{x}_2(0) = v_0 \sin\theta$$

$x_i(0)$ means $x_i(t=0)$

2. Force given by gravity is constant (gravity field):

$$F_1 = m\ddot{x}_1 = 0, \qquad F_2 = -mg = m\ddot{x}_2$$

3. Estimate $x_i(\mathrm{t}) = ?$

$$x_i(\mathrm{t}) = x_i(0) + \int_0^t \frac{dx_i}{dt}dt = \mathrm{x_i}(0) + \int_0^t \dot{x}_i dt$$

$$\dot{x}_i(t) = \dot{x}_i(0) + \int_0^t \frac{d\dot{x}_i}{dt}dt$$

$$\dot{x}_1(t) = \dot{x}_1(t=0) + \int_0^t \ddot{x}_1 dt = v_0\cos\theta + 0$$

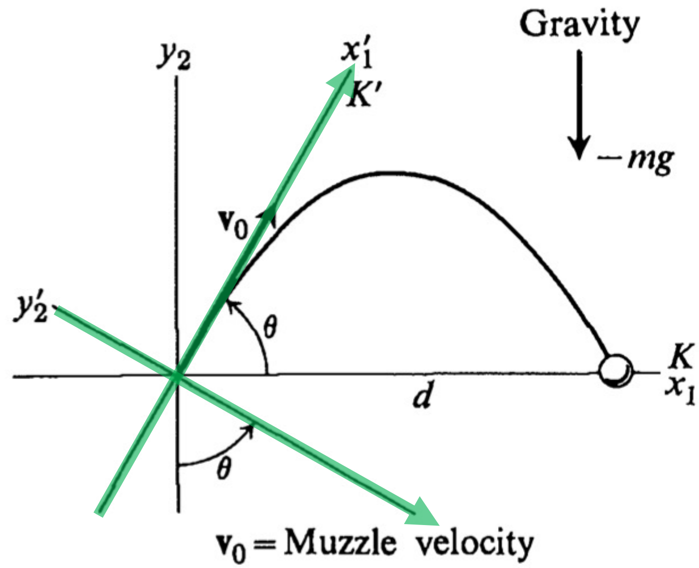
$$\dot{x}_2(t) = \dot{x}_2(t=0) + \int_0^t \ddot{x}_2 dt = v_0\sin\theta + \int_0^t -g dt = v_0 \sin\theta - gt$$

$$x_1(t) = \int_0^t v_0\cos\theta\, dt = v_0 t\cos\theta$$

$$x_2(t) = \int_0^t (v_0\sin\theta - gt)dt = v_0 t\sin\theta - \frac{1}{2}gt^2$$

Newton's second law

$$F_i(t) = m\ddot{x}_i(t)$$



Let's use K' coordinate system

1. Initial condition in terms of location (x_i) and velocity ($\dot{x}_i$):

$$x_i'(t=0) = 0, \qquad \text{with } i = 1,2$$
$$\dot{x}_1'(0) = v_0$$
$$\dot{x}_2'(0) = 0$$

2. Force given by gravity is constant (gravity field):

$$F_1 = m\ddot{x}_1 = -mg\sin\theta\,, \qquad F_2 = -mg\cos\theta = m\ddot{x}_2$$

3. Estimate $x_i(\mathrm{t})$ =?

$$x_i(\mathrm{t}) = x_i(0) + \int_0^t \frac{dx_i}{dt}dt = \mathrm{x_i}(0) + \int_0^t \dot{x}_i dt$$

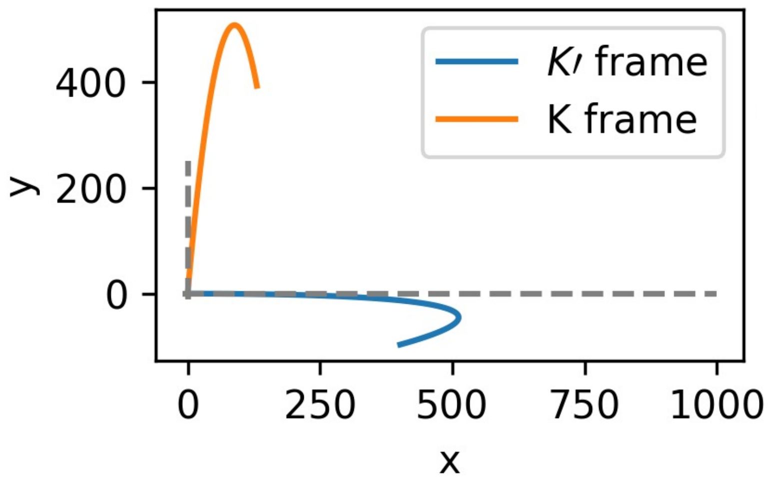
$$\dot{x}_i(t) = \dot{x}_i(0) + \int_0^t \ddot{x}_i dt$$

$$\dot{x}_1(t) = \dot{x}_1(0) + \int_0^t \ddot{x}_1 dt = v_0 - \int_0^t g\sin\theta\, dt = v_0 - gt\sin\theta$$

$$\dot{x}_2(t) = \dot{x}_2(0) + \int_0^t \ddot{x}_2 dt = 0 - \int_0^t g\cos\theta\, dt = -gt\cos\theta$$

$$x_1(t) = \int_0^t (v_0 - gt\sin\theta)dt = v_0 t - \frac{1}{2}gt^2\sin\theta$$

$$x_2(t) = \int_0^t -gt\cos\theta\, dt = -\frac{1}{2}gt^2\cos\theta$$



400
200
0
y
0
250
500
750
1000
x
K' frame
K frame