Identity matrix

- $\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{A} \cdot \boldsymbol{I} = \boldsymbol{A}, \quad \boldsymbol{I} \cdot \boldsymbol{A} = \boldsymbol{A}$

- Index 표기법을 사용하면 identity matrix $\boldsymbol{I}$가 Kronecker delta δ_{ij} 로 표기될 수 있다.

$$\delta_{ij} = 1 \text{ if } i = j$$

$$\delta_{ij} = 0 \text{ if } i \neq j$$

Kronecker delta examples

$$a_{ij}\delta_{ij} = \sum_i \sum_j a_{ij}\delta_{ij} = a_{11}\delta_{11} + a_{12}\delta_{12} \ldots$$

$$= a_{11} + a_{22} + a_{33}$$

$$= \sum_i a_{ii}$$

$$= a_{ii}$$

Notie the dummy index i, so that

$$a_{ii} = a_{jj} = a_{kk} \ldots$$

Q1) $a_{ij} = b_{kk}\delta_{ij}$

1. There are on dummy index, k and two free indicies i, j
2. Therefore, the above means 9 equations.
3. If we expand the dummy index k, we have

$$a_{ij} = (b_{11} + b_{22} + b_{33})\delta_{ij}$$

Kronecker delta

Q2) $\mathbb{E}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$, expand the equation to find explicit component $\mathbb{E}_{1233}$

1. There is no dummy and four free indicies exist, namely, i, j, k, l
2. $\mathbb{E}_{1233} = \lambda\delta_{12}\delta_{33} + \mu(\delta_{13}\delta_{23} + \delta_{13}\delta_{23}) = 0$

Q3) Expand the equation to find explicit component $\mathbb{E}_{2233} = \lambda\delta_{22}\delta_{33} + \mu(\delta_{23}\delta_{23} + \delta_{23}\delta_{23}) = \lambda$

Q4) Expand the equation to find explicit component $\mathbb{E}_{1212} = \lambda\delta_{12}\delta_{12} + \mu(\delta_{11}\delta_{22} + \delta_{12}\delta_{21}) = \mu$

Transpose

- $\boldsymbol{A} = \begin{bmatrix} 3 & 4 & 6 \\ -3 & 2 & 5 \\ 1 & -1 & -4 \end{bmatrix}$

- $\boldsymbol{A}^T = \begin{bmatrix} 3 & -3 & 1 \\ 4 & 2 & -1 \\ 6 & 5 & -4 \end{bmatrix}$

- In tensor notation, $A_{ij}^T = A_{ji}$

Matrix addition and dot, double-dot products

- Addition
  - $\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$
  - $C_{ij} = A_{ij} + B_{ij}$
- Dot products
  - $\boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{B}$
  - $C_{ij} = A_{ik}B_{kj}$ (Find the free and non-free indices!)
  - Multiplication is not commutative
    - $\boldsymbol{A} \cdot \boldsymbol{B} \neq \boldsymbol{B} \cdot \boldsymbol{A}$
- Double dot products
  - $d = \boldsymbol{A}:\boldsymbol{B}$ (denote d is a scalar quantity thus is **not** denoted in bold-face)
  - $d = A_{ij}B_{ij}$

예제

두 행렬이 다음과 같이 주어져 있다.

$$\boldsymbol{A} = \begin{bmatrix} 3 & 4 & 6 \\ -3 & 2 & 5 \\ 1 & -1 & -4 \end{bmatrix}$$

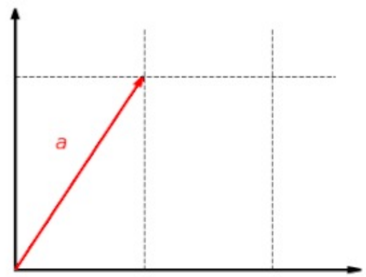
$$\boldsymbol{B} = \begin{bmatrix} 5 & 2 & 9 \\ 0 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

두 행렬의 내적의 결과를 $\boldsymbol{C}$ 라 할 때 ($\boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{B}$)

- C_{13} 값은 무엇인가?
- C_{21} 값은 무엇인가?

Linear transformation, Linear map, Linear operation (선형변환)

- Let's understand 'linear transformation' as a function that inputs vectors and output vectors



$f(\boldsymbol{a}) = f([a_x, a_y])$

Input: vector $\boldsymbol{a}$
Output: $f(\boldsymbol{a})$

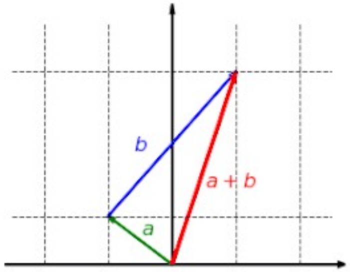
Input: (a_x, a_y)
Output: $(2a_x, a_y)$

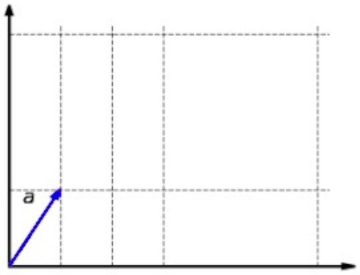
What this linear map does:
takes in a vector, multiplies 2 only on a_x and then puts out $(2a_x, a_y)$

A linear map should satisfy two conditions:
1) Additivity, 2) Homogeneity of degree 1

$f(\boldsymbol{a}+\boldsymbol{b}) = f(\boldsymbol{a}) + f(\boldsymbol{b})$

$f(\lambda\boldsymbol{a}) = \lambda f(\boldsymbol{a})$

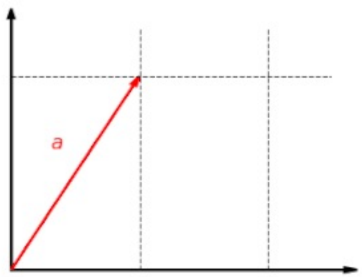




https://en.wikipedia.org/wiki/Linear_map

Linear transformation, Linear map, Linear operation (선형변환)

- Let's understand 'linear transformation' as a function that inputs vectors and output vectors



$$\begin{bmatrix} 2a_x \\ a_y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \end{bmatrix}$$

$$f(\boldsymbol{a}) = f([a_x, a_y])$$

What this linear map does:
takes in a vector, multiplies 2 only on a_x and then puts out $(2a_x, a_y)$

$$f(\boldsymbol{a}) = \boldsymbol{A} \cdot \boldsymbol{x} = A_{ij} x_j$$

$$\boldsymbol{A} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

- rotation
  - by 90 degrees counterclockwise:

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

  - by an angle θ counterclockwise:

$$\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

- scaling by 2 in all directions:

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2\mathbf{I}$$

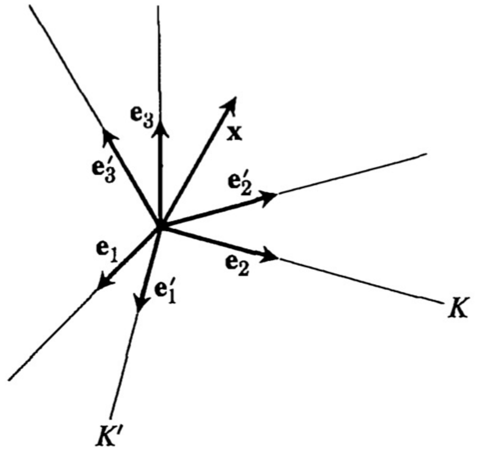
- horizontal shear mapping:

$$\mathbf{A} = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$$

https://en.wikipedia.org/wiki/Linear_map

Rotation (transformation) of the coordinate system

Relationship between the components of a unit vector expressed with respect to two different Cartesian bases with the same origin (not necessarily orthonormal);



Two cartesian coordinates (K and K') with two separate sets of basis vectors ($\mathbf{e}_i$ and $\mathbf{e}'_i$) and a vector $\mathbf{x}$

Any vector $\mathbf{x}$ can be resolved into components with respect to either the K or the K' system.

$$\mathbf{x} = (\mathbf{x} \cdot \mathbf{e}_j)\mathbf{e}_j = x_j \mathbf{e}_j$$

If we take $\mathbf{x} = \mathbf{e}'_i$ (a certain basis vector of K')

$$\mathbf{e}'_i = (\mathbf{e}'_i \cdot \mathbf{e}_j)\mathbf{e}_j \equiv a_{ij}\mathbf{e}_j$$

The nine terms a_{ij} (for each of three basis vectors; $i = 1$, $i = 2$, and $i = 3$) are directional cosines of the angles between the six axes:

$$\boldsymbol{R} \equiv (a_{ij}) \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$\boldsymbol{R}$ is known as the transformation matrix (or rotation matrix) in three dimension.

Rotation (transformation) of the coordinate system

$\mathbf{e}'_i \equiv a_{ij}\mathbf{e}_j$ Switching $j \to k$ $\mathbf{e}'_i \equiv a_{ik}\mathbf{e}_k$

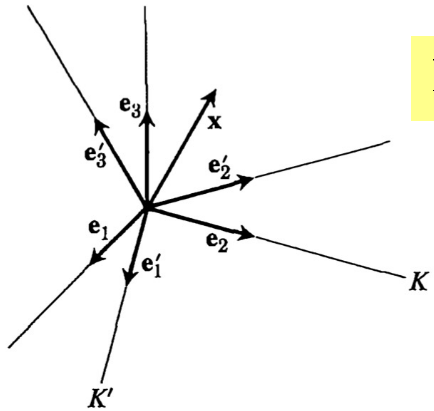
Earlier, we defined: $a_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j$

And $a_{ij}\mathbf{e}_j = \mathbf{e}'_i \cdot \mathbf{e}_j \cdot \mathbf{e}_j$

$\therefore a_{ij}\,\mathbf{e}_j = \mathbf{e}'_i|\mathbf{e}| \to a_{ij}\mathbf{e}_j = \mathbf{e}'_i$

$\delta_{ij} = \mathbf{e}'_i \cdot \mathbf{e}'_j = a_{ik}\mathbf{e}_k \cdot \mathbf{e}'_j = a_{ik}b_{kj}$ $\quad [a] = [b]^{-1}$

If we defined: $b_{ij} = \mathbf{e}_i \cdot \mathbf{e}'_j$



Two cartesian coordinates (K and K') with two separate sets of basis vectors ($\mathbf{e}_i$ and $\mathbf{e}'_i$) and a vector $\mathbf{x}$

Any vector $\mathbf{x}$ may be expressed in the K system as

$$\mathbf{x} = x_j\mathbf{e}_j$$

or as in the K' system using primed basis such as

$$\mathbf{x} = x'_i\mathbf{e}'_i$$

They are the same vector so one can equate

$$\mathbf{x} = x'_i\mathbf{e}'_i = x_j\mathbf{e}_j$$

One could replace $\mathbf{e}'_i$ with $a_{ij}\mathbf{e}_j$

$$x'_i a_{ij}\mathbf{e}_j = x_j\mathbf{e}_j$$

$$x_j = a_{ji}x'_i$$

so that

$$x_j = x'_i a_{ij} \quad [x_j]^T = [x'_i a_{ij}]^T \quad x_j = [a_{ij}]^T[x'_i]^T$$

Or equivalently, swapping the indices i and j gives:

$$x_i = a_{ij}x'_j$$

Examples 2D

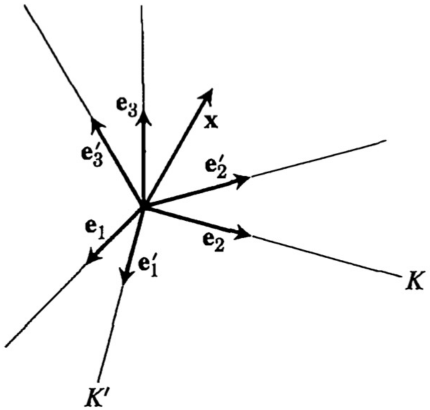
- $\mathbf{e}_1, \mathbf{e}_2$ basis vector가 $(1,0), (0,1)$로 표현되고 반면, $\mathbf{e}'_1, \mathbf{e}'_2$ 벡터가 각각 $(0,-1), (1,0)$ 으로 주어져 있다. 주어진 네 basis vector의 구성성분 값들이 $\mathbf{e}_1, \mathbf{e}_2$ basis vector로 이루어진 좌표계 K에 대해 참조되어 있으며, $\mathbf{e}'_1, \mathbf{e}'_2$로 이루어진 좌표계를 K'이라 하자.

- 이때 벡터 $\boldsymbol{a}$가 좌표계K에 대해 참조되어 다음과 같이 구성성분값들이 주어져 있다고 한다.

$$\boldsymbol{a} = (2,5)$$

Q) 벡터 $\boldsymbol{a}$의 구성성분 값들을 좌표계 K'에 대해 구하시오.

Examples 3D



- $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ basis vector가 $(1,0,0), (0,1,0), (0,0,1)$로 표현되고 반면, $\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'$ 벡터가 각각 $(0,-1,0), (0,0,1), (-1,0,0)$ 으로 주어져 있다. 주어진 여섯 basis vector의 구성성분 값들이 $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ basis vector로 이루어진 좌표계 K에 대해 참조되어 있으며, $\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'$으로 이루어진 좌표계를 K'이라 하자.

- 이때 벡터 $\boldsymbol{a}$가 좌표계K에 대해 참조되어 다음과 같이 구성성분값들이 주어져 있다고 한다.

$$\boldsymbol{a} = (1,1,1)$$

Q) 벡터 $\boldsymbol{a}$의 구성성분 값들을 좌표계 K'에 대해 구하시오.