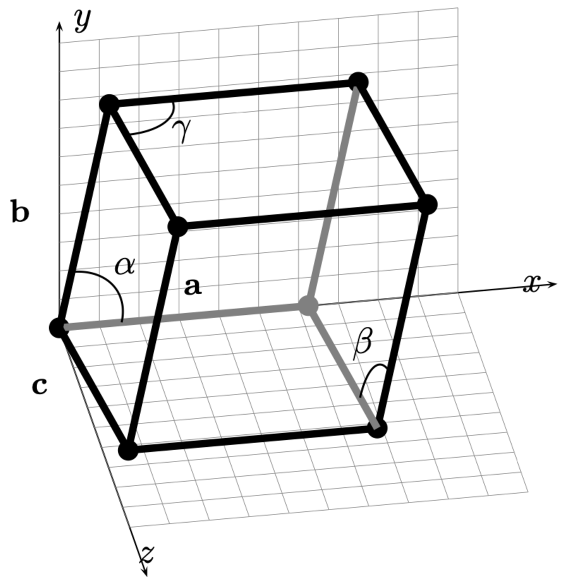
Convert from Cartesian to Voigt

- $\mathbb{E}_{11}^{(\mathrm{voigt})} = \mathbb{E}_{1111}^{(\mathrm{cartesian})}, \mathbb{E}_{23}^{(\mathrm{voigt})} = \mathbb{E}_{2233}^{(\mathrm{cartesian})}, \mathbb{E}_{41}^{(\mathrm{voigt})} = \mathbb{E}_{2311}^{(\mathrm{cartesian})}$

- Material anisotropy
- Symmetry can be represented by transformation matrix
- $\boldsymbol{Q} = Q_{ij}\mathbf{e}_\mathrm{i} \otimes \mathbf{e}_\mathrm{j}$, such that $\boldsymbol{Q}^{-1} = \boldsymbol{Q}^T$
- The invariance of the stiffness tensor under these transformations (due to symmetry) is:
- $\mathbb{E}^{(new)} = \boldsymbol{Q} \cdot \boldsymbol{Q} \cdot \mathbb{E}^{(old)} \cdot \boldsymbol{Q}^T \cdot \boldsymbol{Q}^T$
- Due to symmetry the resulting tensor should be equivalent with the original one: $\mathbb{E}^{(new)} \equiv \mathbb{E}^{(old)}$

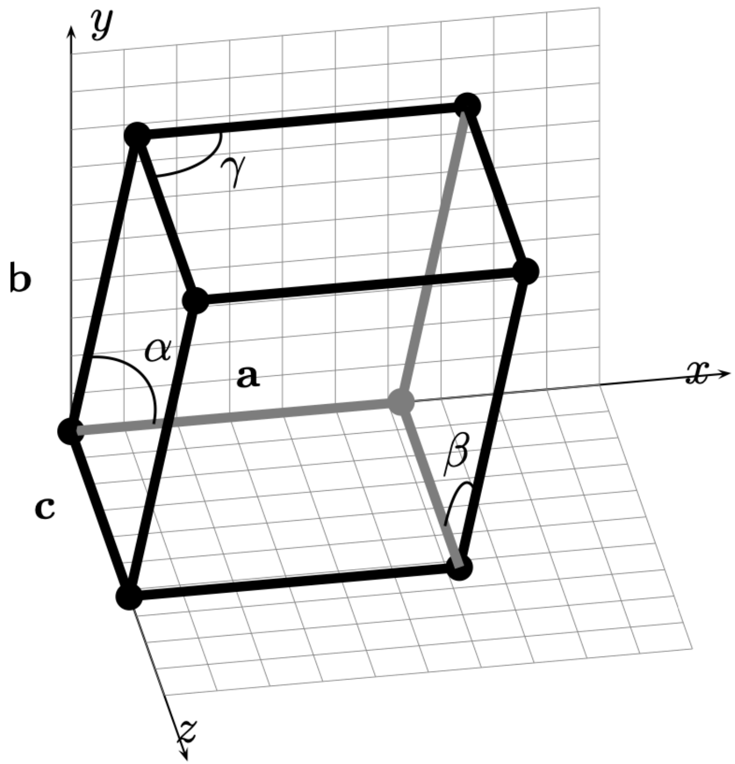
Triclinic (no symmetry)



Triclinic: no symmetry planes, fully anisotropic.
$\alpha, \beta, \gamma < 90$
Number of independent coefficients: 21
Symmetry transformation: None

$$C = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ & & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ & & & C_{2323} & C_{2313} & C_{2312} \\ & symm & & & C_{1313} & C_{1312} \\ & & & & & C_{1212} \end{bmatrix}$$

Monoclinic (one symmetry plane)



Monoclinic: one symmetry plane (xy).
$a \neq b \neq c,\ \beta = \gamma = 90, \alpha < 90$
Number of independent coefficients: 13
Symmetry transformation: reflection about z-axis

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & C_{1112} \\ & C_{2222} & C_{2233} & 0 & 0 & C_{2212} \\ & & C_{3333} & 0 & 0 & C_{3312} \\ & & & C_{2323} & C_{2313} & 0 \\ & symm & & & C_{1313} & 0 \\ & & & & & C_{1212} \end{bmatrix}$$

Monoclinic (one symmetry plane)

- For the case of Monoclinic:
- $\boldsymbol{Q} = Q_{ij}\mathbf{e}_\mathrm{i} \otimes \mathbf{e}_\mathrm{j} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
- Let’s take a look at the invariance due to symmetry
- $\mathbb{E}^{(new)} = \boldsymbol{Q} \cdot \boldsymbol{Q} \cdot \mathbb{E}^{(old)} \cdot \boldsymbol{Q}^T \cdot \boldsymbol{Q}^T$

due to symmetry the resulting tensor should be equivalent with the original one:
$\mathbb{E}^{(new)} \equiv \mathbb{E}^{(old)}$

- In its matrix form:
  - $\mathbb{E}_{ijkl}^{(new)} = Q_{im}Q_{jn}Q_{ko}Q_{lp}\mathbb{E}_{mnop}^{(old)}$

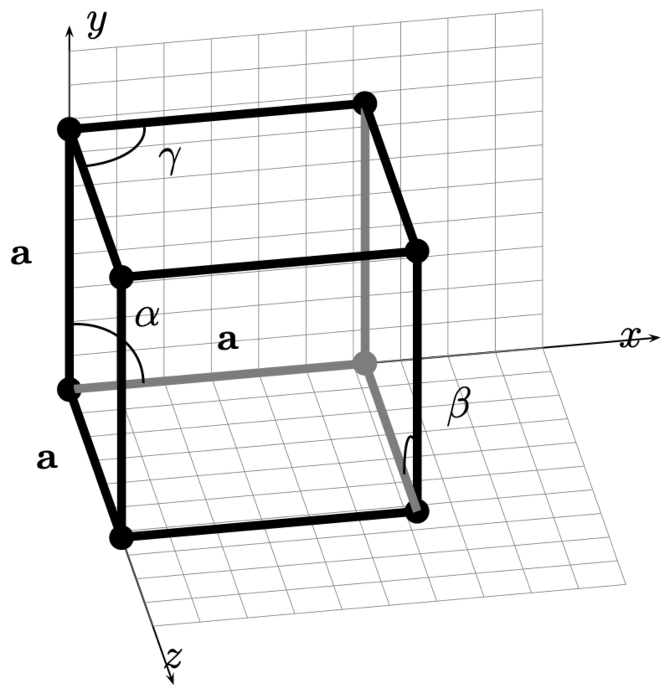
monoclinic (one symmetry plane)

- For the case of Monoclinic:
- $\boldsymbol{Q} = Q_{ij}\mathbf{e}_\mathrm{i} \otimes \mathbf{e}_\mathrm{j} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
  - $\mathbb{E}_{ijkl}^{(new)} = Q_{im}Q_{jn}Q_{ko}Q_{lp}\mathbb{E}_{mnop}^{(old)}$
  - Ex: $\mathbb{E}_{11}^{(voigt)} = \mathbb{E}_{1111} = Q_{1m}Q_{1n}Q_{1o}Q_{1p}\mathbb{E}_{mnop}^{(old)}$
    - If you look at the matrix form of symmetry operator Q in the above, only diagonal components are non-zero. Therefore, $Q_{ij} = 0$ if $i \neq j$.
    - $\mathbb{E}_{11}^{(voigt)} = \mathbb{E}_{1111} = Q_{11}Q_{11}Q_{11}Q_{11}\mathbb{E}_{1111}^{(old)} = \mathbb{E}_{1111}^{(old)}$
    - Therefore, $\mathbb{E}_{11}^{(voigt)} = \mathbb{E}_{1111}$

monoclinic (one symmetry plane)

- For the case of Monoclinic:
- $\boldsymbol{Q} = Q_{ij}\mathbf{e}_\mathrm{i} \otimes \mathbf{e}_\mathrm{j} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
  - $\mathbb{E}_{ijkl}^{(new)} = Q_{im}Q_{jn}Q_{ko}Q_{lp}\mathbb{E}_{mnop}^{(old)}$
  - Ex: $\mathbb{E}_{14}^{(voigt)} = \mathbb{E}_{1123} = Q_{1m}Q_{1n}Q_{2o}Q_{3p}\mathbb{E}_{mnop}^{(old)} = Q_{11}Q_{11}Q_{22}Q_{33}\mathbb{E}_{1123}^{(old)} = 1\times1\times1\times(-1)\times\mathbb{E}_{1123}^{(old)} = -\mathbb{E}_{1123}^{(old)}$
    - Therefore, in order to satisfy $\mathbb{E}_{1123} = -\mathbb{E}_{1123}$, $\mathbb{E}_{1123}$ should be zero.
    - Which means $\mathbb{E}_{14}^{(voigt)}$ should be zero

Cubic



Cubic: three mutually orthogonal planes of reflection symmetry plus 90° rotation symmetry with respect to those planes. $a = b = c$, $\alpha = \beta = \gamma = 90$
Number of independent coefficients: 3
Symmetry transformations: reflections and 90° rotations about all three orthogonal planes

$$Q = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \; Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} C_{1111} & C_{1122} & C_{1122} & 0 & 0 & 0 \\ & C_{1111} & C_{1122} & 0 & 0 & 0 \\ & & C_{1111} & 0 & 0 & 0 \\ & & & C_{1212} & 0 & 0 \\ & symm & & & C_{1212} & 0 \\ & & & & & C_{1212} \end{bmatrix}$$

References and acknowledgements

- References
  - An introduction to Continuum Mechanics – M. E. Gurtin
  - Metal Forming – W.F. Hosford, R. M. Caddell (번역판: 금속 소성 가공 - 허무영)
  - Fundamentals of metal forming (R. H. Wagoner, J-L Chenot)
  - http://www.continuummechanics.org (very good on-line reference)
- Acknowledgements
  - Some images presented in this lecture materials were collected from Wikipedia.