Symmetries

- $\sigma_{ij} = \sigma_{ji}$ gives $\mathbb{E}_{ijkl} = \mathbb{E}_{jikl}$ thus, the required number of elastic constants reduces from 3x3x3x3 to 6x3x3
- Similarly, $\varepsilon_{ij} = \varepsilon_{ji}$ gives $\mathbb{E}_{ijkl} = \mathbb{E}_{ijlk}$ so that we have the required of number of constants 6x6=36
- The required number of constants can be further reduced. Consider the elastic energy:
- $\phi = \int \sigma_{ij} d\varepsilon_{ij}$
- $\sigma_{ij} = \frac{\partial \phi}{\partial \varepsilon_{ij}} = \mathbb{E}_{ijkl}\varepsilon_{kl}$
- If we apply partial derivative once again, we have
- $\frac{\partial^2 \phi}{\partial \varepsilon_{mn} \partial \varepsilon_{ij}} = \frac{\partial}{\partial \varepsilon_{mn}}\left(\mathbb{E}_{ijkl}\varepsilon_{kl}\right)$ since $\mathbb{E}$ is ‘constant’, we have
- $\frac{\partial^2 \phi}{\partial \varepsilon_{mn} \partial \varepsilon_{ij}} = \mathbb{E}_{ijkl}\left(\frac{\partial \varepsilon_{kl}}{\partial \varepsilon_{mn}}\right) = \mathbb{E}_{ijkl}\delta_{km}\delta_{ln} = \mathbb{E}_{ijmn}$

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- We could do the 2nd order derivative in a different way (say, instead of $\frac{\partial^2 \phi}{\partial \varepsilon_{mn} \partial \varepsilon_{ij}}$ we could have done $\frac{\partial^2 \phi}{\partial \varepsilon_{ij} \partial \varepsilon_{mn}} = \frac{\partial}{\partial \varepsilon_{ij}}\left(\frac{\partial \phi}{\partial \varepsilon_{mn}}\right) = \frac{\partial}{\partial \varepsilon_{ij}}\left(\frac{\partial \phi}{\partial \varepsilon_{mn}}\right) = \mathbb{E}_{mnij}$
- The two cases (regardless of the order of derivative) should give equivalent result so that
- $\mathbb{E}_{ijmn} = \mathbb{E}_{mnij}$

- This summarizes our finding on the symmetries in elastic tensor:

How many constants are required?

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix}\mathbb{E}_{1111} & \mathbb{E}_{1122} & \mathbb{E}_{1133} & \mathbb{E}_{1123} & \mathbb{E}_{1113} & \mathbb{E}_{1112}\\ \mathbb{E}_{2211} & \mathbb{E}_{2222} & \mathbb{E}_{2233} & \mathbb{E}_{2223} & \mathbb{E}_{2213} & \mathbb{E}_{2212}\\ \mathbb{E}_{3311} & \mathbb{E}_{3322} & \mathbb{E}_{3333} & \mathbb{E}_{3323} & \mathbb{E}_{3313} & \mathbb{E}_{3312}\\ \mathbb{E}_{2311} & \mathbb{E}_{2322} & \mathbb{E}_{2333} & \mathbb{E}_{2323} & \mathbb{E}_{2313} & \mathbb{E}_{2312}\\ \mathbb{E}_{1311} & \mathbb{E}_{1322} & \mathbb{E}_{1333} & \mathbb{E}_{1323} & \mathbb{E}_{1313} & \mathbb{E}_{1312}\\ \mathbb{E}_{1211} & \mathbb{E}_{1222} & \mathbb{E}_{1233} & \mathbb{E}_{1223} & \mathbb{E}_{1213} & \mathbb{E}_{1212}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{12}\end{bmatrix}$$

How many constants do we need?

If the coordinate system happens to give strain and stress all principal values:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{bmatrix} = \begin{bmatrix} \mathbb{E}_{1111} & \mathbb{E}_{1122} & \mathbb{E}_{1133} \\ & \mathbb{E}_{2222} & \mathbb{E}_{2223} \\ & & \mathbb{E}_{3333} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \end{bmatrix}$$

example

- Fe(1-0.025)-Al(0.025) alloy의 탄성 계수는 다음과 같이 주어진다.

- $\mathbb{E}_{11} = 270.71$, $\mathbb{E}_{12} = 128.03$, $\mathbb{E}_{44} = 108.77$
- Fe-Al alloy는 Body-centered cubic 결정 구조를 가지고, 결정 대칭성에 의해 다음과 같은 탄성 거동을 한다.

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{bmatrix} = \begin{bmatrix}\mathbb{E}_{11} & \mathbb{E}_{12} & \mathbb{E}_{13} & 0 & 0 & 0\\ \mathbb{E}_{21} & \mathbb{E}_{22} & \mathbb{E}_{23} & 0 & 0 & 0\\ \mathbb{E}_{31} & \mathbb{E}_{32} & \mathbb{E}_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & \mathbb{E}_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & \mathbb{E}_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & \mathbb{E}_{66}\end{bmatrix}\begin{bmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_3\\ \epsilon_4\\ \epsilon_5\\ \epsilon_6\end{bmatrix}$$

- 뿐만 아니라, cubic 결정구조의 대칭성으로 인해 $\mathbb{E}_{11} = \mathbb{E}_{22} = \mathbb{E}_{33}, \mathbb{E}_{44} = \mathbb{E}_{55} = \mathbb{E}_{66}, \mathbb{E}_{12} = \mathbb{E}_{13} = \mathbb{E}_{23}$

Example

- Fe(1-0.025)-Al(0.025) alloy의 단결정에 다음과 같은 탄성 변형률이 나타나기 위해 필요한 응력 상태는?

$$\begin{bmatrix} 0.0001 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$